Math 112 lecture for Friday, Week 11

Continuity and derivatives

(Supplemental reading: Sections 5.1, 6.1, and 6.2 in Swanson.)

Our results will hold for real and complex functions unless noted. To treat both cases simultaneously, let $F = \mathbb{R}$ or $\mathbb{C}$.

New-from-old limit theorems.

Functions can sometimes be decomposed into sums and products of simpler functions. Taking a cue from what we did earlier with sequences, we first find limits of some simple functions using ε-δ proofs, and then we prove a general result that allows us to compute limits of more complicated functions built from these without resorting to ε-δ proofs.

Recall the definition of a limit of a function:

Definition. Let $A \subseteq F$ and $f\colon A \to F$. Let $a \in F$ be a limit point of A. Then the *limit of $f(x)$ as x approaches a is $L \in F$* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then

$$|f(x) - L| < \varepsilon.$$

If the limit is L, then we write $\lim_{x\to a} f(x) = L$.

Now we compute the limits of a couple of simple functions. These, combined with the "new-from-old" limit theorem we prove next, will allow us to easily compute the limits of all rational functions, i.e., of all quotients of polynomials.

Proposition. Let $a, c \in F$. Then

(a) $\lim\limits_{x\to a} c = c$.

(b) $\lim\limits_{x\to a} x = a$.

Proof.

(a) In this case, our function is $f(x) = c$ for all x, and we are claiming the limit is c. Given $\varepsilon > 0$,

$$|f(x) - c| = |c - c| = 0 < \varepsilon$$

for *all* x. This means the ε-δ definition of the limit is satisfied for any choice of $\delta > 0$, reflecting that fact that it is pretty easy to make $f(x)$ close to c!

(b) Here, $f(x) = x$ for all x, and we claim the limit is a. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Then, if we assume $0 < |x - a| < \delta = \varepsilon$, it follows that

$$|f(x) - a| = |x - a| < \varepsilon.$$

□

Here is the function-version of a "new-from-old" limit theorem, analogous to the one given earlier for sequences. It allows us to determine limits of functions that are built from simpler functions.

Theorem. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then,

(a) $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$,

(b) $\lim_{x \to a} f(x)g(x) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x)) = LM$,

(c)
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M},$$
provided $\lim_{x \to a} g(x) \neq 0$.

Proof. The proof of this theorem is almost identical to the proof of the limit theorems for sequences given earlier.

(a) Given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2}.$$

(Again, as in the earlier proof for sequences, at first we might find a δ_1 for f and a δ_2 for g, then let $\delta := \min\{\delta_1, \delta_2\}$ to work simultaneously for both f and g.) It then follows that for $0 < |x - a| < \delta$,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

We leave the proofs of parts (b) and (c) these exercises. □

Example. Claim: $\lim_{x\to 5} x^2 = 25$.

Proof. From the earlier Proposition, we have $\lim_{x\to 5} x = 5$. Using our limit theorem,

$$\lim_{x\to 5} x^2 = \lim_{x\to a}(x \cdot x) = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) = 5 \cdot 5 = 25.$$

Compare how easy this proof was compared to the earlier ε-δ proof in which we chose $\delta = \min\{1, \varepsilon/11\}$.

Example. Claim: $\lim_{x\to 1} \dfrac{x^2+5}{x^3-3x+1} = -6$.

Proof.

$$\begin{aligned}\lim_{x\to 1} \frac{x^2+5}{x^3-3x+1} &= \frac{\lim_{x\to 1}(x^2+5)}{\lim_{x\to 1}(x^3-3x+1)} \\ &= \frac{(\lim_{x\to 1} x)^2 + \lim_{x\to 1} 5}{(\lim_{x\to 1} x)^3 + \lim_{x\to 1}(-3)\lim_{x\to 1} x + \lim_{x\to 1} 1} \\ &= \frac{1^2+5}{1^3 - 3\cdot 1 + 1} \\ &= -6.\end{aligned}$$

□

Continuity.

In the previous example, we found the limit of $f(x) = \frac{x^2+5}{x^3-3x+1}$ at $x = 1$ to simply be $f(1)$. A similar proof would show that for any $a \in F$, we have $\lim_{x\to a} f(x) = f(a)$ (for this particular function f). This function is an example of a *continuous* function, where we can find the value of the limit simply by evaluating the function at the limit point. If all functions were continuous, there would be little need for the notion of a limit. Limits are more interesting when you need to determine the value of a function at a point where the function is undefined,

Definition. Let $A \subseteq F$ and $f\colon A \to F$. Then f *is continuous at* $a \in A$ if for all $\varepsilon > 0$ there exist $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. We say f is *continuous* or *continous on* A if f is continuous at every point of A.

Remarks.

(a) In comparing the definition of limits with that of continuity, note that in the latter, we have the condition $|x - a| < \delta$ rather than $0 < |x - a| < \delta$ and the condition $|f(x) - f(a)| < \varepsilon$ rather than $|f(x) - L| < \varepsilon$. This says that if every point of A is a limit point of A, for instance if A is an open or closed ball, then f is continuous on A if and only if

$$\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a)$$

for all $a \in A$. (Thus, we sometimes say the continuous functions are those that commute with limits.) In other words, to find the limit of $f(x)$ as x approaches a, we simply evaluate f at a. For instance, every polynomial or quotient of polynomials, every trig function, the exponential function, logarithms, the square root function, and the absolute value function are continuous wherever they defined.

On the other hand, if $a \in A$ is not a limit point of A, the definition of continuity implies that f is automatically continuous at a. We leave this as an exercise. A close look at the definition of limits reveals that limits are not defined for points that are not limit points of the domain.

(b) Using the limit theorems, is easy to show that sums, products and quotients (where defined) of continuous functions are continuous.

(c) A straightforward argument from the definitions shows that compositions of continuous functions are continuous.

Derivatives.

Definition. Let $A \subseteq F$, and let $a \in A$ be a limit point of A. Let $f\colon A \to F$. Then the *derivative of f at a* is

$$\frac{df}{dx}(a) := f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If the limit exists, we say f is *differentiable* at a.

Remark. Equivalently, we could define the derivative at a to be

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^2$. Then $f'(i) = 2i$.

Proof. Calculate:

$$f'(i) = \lim_{h \to 0} \frac{f(i+h) - f(i)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(i+h)^2 - i^2}{h} \\
&= \lim_{h\to 0} \frac{(i^2 + 2ih + h^2) - i^2}{h} \\
&= \lim_{h\to 0} \frac{2ih + h^2}{h} \\
&= \lim_{h\to 0} (2i + h) \\
&= 2i.
\end{aligned}
$$

□

The usual properties of derivatives hold over $\mathbb{C}$ as well as $\mathbb{R}$:

$$(z^n)' = nz^{n-1}, \quad (f+g)' = f' + g', \quad (cf)' = c(f') \text{ for a constant } c,$$

including the product and quotient rules:

$$(fg)' = f'g + fg', \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

and the chain rule:

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proposition. Differentiable functions are continuous. Suppose $f\colon A \to F$ is differentiable, and let a be a limit point of A. Then f is continuous at a.

Proof. Using our limit theorems, we have

$$
\begin{aligned}
0 &= f'(a) \cdot 0 \\
&= \left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right)\left(\lim_{x\to a}(x - a)\right) \\
&= \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) \\
&= \lim_{x\to a} \left(f(x) - f(a)\right).
\end{aligned}
$$

Thus, $\lim_{x\to a}(f(x)-f(a)) = 0$. We now us the facts that $\lim_{x\to a}(f(x)-f(a))$ and $\lim_{x\to a} f(a)$ exist along with the sum formula for limits to see that $\lim_{x\to a} f(x)$ exists and

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a) = 0 + f(a) = f(a).$$

So $\lim_{x\to a} f(x) = f(a)$, as required. □

Proposition (product rule). Suppose that f and g are differentiable as some point $a \in F$. Then so is their product, fg, and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. We have

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h} && \text{(def. of the derivative)} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h)-f(a)g(a)}{h} && \text{(def. of } fg\text{)} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h} && \text{(tricky!)} \\
&= \lim_{h\to 0} \frac{(f(a+h)-f(a))g(a+h)+f(a)(g(a+h)-g(a))}{h} \\
&= \lim_{h\to 0} \frac{(f(a+h)-f(a))g(a+h)}{h} + \lim_{h\to a} \frac{f(a)(g(a+h)-g(a))}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \lim_{h\to a} g(a+h) + f(a) \lim_{h\to a} \frac{g(a+h)-g(a)}{h} \\
&= f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

The fact that $\lim_{h\to 0} g(a+h) = g(\lim_{h\to a}(a+h)) = g(a)$ follows from the fact that g, being differentiable at a, is continuous at a. □

Corollary (quotient rule). Suppose that f and g are differentiable as some point $a \in F$ and that $g(a) \neq 0$. Then the quotient, f/g is differentiable at a, and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}.$$

Proof. Apply the product rule and the chain rule:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) \\
&= f'(a)\frac{1}{g(a)} + f(a)\left(\frac{1}{g}\right)'(a) \\
&= f'(a)\frac{1}{g(a)} + f(a)\left(-\frac{g'(a)}{g(a)^2}\right) \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.
\end{aligned}$$

□