Math 112 lecture for Wednesday, Week 10

SERIES TESTS I

(Supplemental reading: Section 9.2 in Swanson.)

Our next goal is to present the standard collection of tests for determining whether a series converges.[1] Here is the list:

1. **the geometric series test**
2. **the n-th term test**
3. **the comparison test**
4. the limit comparison test
5. the alternating series test
6. the absolute convergence test
7. the ratio test
8. the root test
9. the integral test
10. the p-series test.

We have already discussed the geometric series test, but we will state it again here for completeness and for review.

1. The geometric series test. Let $r \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$. When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

More generally, for $a \in \mathbb{C}$ and $k \in \mathbb{N}$, if $|r| < 1$, then

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}.$$

Examples. (See earlier in our notes for more examples.)

[1] Note that knowing that a series converges is different from knowing its limit.

(a)

$$\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2}{2-i}\cdot\frac{2+i}{2+i} = \frac{4}{5}+\frac{1}{5}i.$$

(b)

$$\sum_{n=2}^{\infty} 8\left(\frac{2^{n+2}}{5^n}\right) = 8\sum_{n=2}^{\infty}\frac{4\cdot 2^n}{5^n} = 32\sum_{n=2}^{\infty}\left(\frac{2}{5}\right)^n = 32\left(\frac{2}{5}\right)^2\frac{1}{1-\frac{2}{5}} = \frac{128}{15}.$$

(c) $\sum_{n=1}^{\infty}(4i)^n$ diverges since $|4i| = 4 \geq 1$.

2. The n-th term test. The n-th term test is a criterion for divergence of a complex series. It says that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ converges, as we will see with the harmonic series, below.

Proposition. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Proof. To prove this result, we prove its contrapositive[2]: if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. To see this, suppose that $\sum a_n = s$, i.e., $\lim s_n = s$ where $s_n = \sum_{k=1}^{n} a_k$ is the n-th partial sum of $\{a_n\}$. We have that $s_n - s_{n-1} = a_n$. Therefore, using our limit theorems

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

$\square$

Examples.

(a) The series $\sum_{n=1}^{\infty}(-1)^n$ diverges since $\lim_{n\to\infty}(-1)^n$ does not exist. In particular, $\lim_{n\to\infty}(-1)^n \neq 0$.

(b) The series $\sum_{n=1}^{\infty}\frac{3n^2-2n+1}{2n^2+5}$ does not converge since $\lim_{n\to\infty}\frac{3n^2-2n+1}{2n^2+5} = \frac{3}{2} \neq 0$.

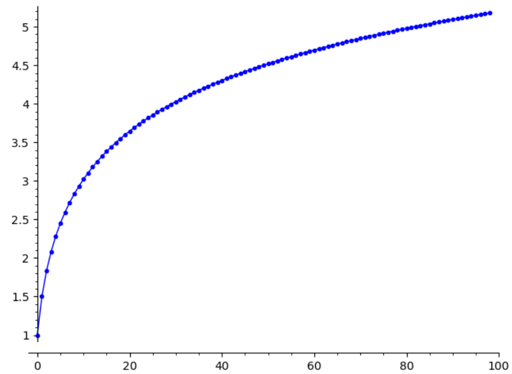
(c) Consider series $\sum_{n=1}^{\infty}\frac{1}{n}$. We have $\lim\frac{1}{n} = 0$, however, as we will see below, this series *diverges*. So *the converse of the proposition does not hold*. The proposition can only be used to prove divergence (not convergence).

The harmonic series. The *harmonic series* is the series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Here is a plot of its partial sums, $1,\ 1+1/2,\ 1+1/2+1/3, \ldots$:

[2]The *contrapositive* of a statement "P implies Q" is "not Q implies not P".



It looks like these partial sums might be converging, but in fact they do not. You may notice that this plot is reminiscent of the graph of the logarithm, which hints that to see substantial growth in the partial sums, we will need to consider partial sums of an exponentially-growing number of terms of the series. The first proof that the harmonic series diverges, which we present below, is due to Nicole Orseme around 1350. The theorem is a special case of the p-series test which we will discuss later and easily prove using the integral test.

Theorem. The harmonic series diverges.

Proof. The first partial sum of the harmonic series is

$$\sum_{n=1}^{1} \frac{1}{n} = 1.$$

The second partial sum is

$$\sum_{n=1}^{2} \frac{1}{n} = 1 + \frac{1}{2}$$

The fourth:

$$\sum_{n=1}^{2^2} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}.$$

The eighth:

$$\begin{aligned}
\sum_{n=1}^{2^3} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\
&= 1 + 3 \cdot \frac{1}{2}.
\end{aligned}$$

An induction proof shows that

$$s_{2^k} = \sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + k \cdot \frac{1}{2}.$$

for all $k \geq 0$. Thus, the sequence of partial sums for the harmonic series is unbounded and, hence, diverges. $\square$

3. The comparison test. In the following proposition, notice that the test only applies to *nonnegative real sequences*. Also, notice the crucial role of the monotone convergence theorem in the proof. (Recall that MCT says that if a real sequence is monotone and bounded above, then it converges.)

Proposition (series comparison test). Suppose that $\{a_n\}$ and $\{b_n\}$ are real sequences with

$$0 \leq a_n \leq b_n$$

for all n. Then

(a) $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

(b) $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

Proof. Let

$$\begin{aligned}
s_n &= a_1 + a_2 + \cdots + a_n \\
t_n &= b_1 + b_2 + \cdots + b_n
\end{aligned}$$

be the respective partial sums. Since $a_n \leq b_n$ for all n, we have $s_n \leq t_n$ for all n. Since both sequences have nonnegative terms, their sequences of partial sums are monotone increasing. To prove part (a), suppose that $\sum b_n$ converges, and say $\sum b_n = t$. This means that $\lim t_n = t$. By the monotone convergence theorem, we know that $t = \sup\{t_n\}$. Thus,

$$s_n \leq t_n \leq t = \sup\{t_n\}.$$

Hence, $\{s_n\}$ is both monotone increasing and bounded above. Again by the monotone convergence theorem, $\{s_n\}$ converges, i.e., $\sum a_n$ converges.

Part (b) is the contrapositive of part (a) and, hence, follows immediately. $\square$

Examples.

(a) For all $n \geq 1$

$$0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}.$$

We showed in an earlier lecture that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. Thus, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

converges. It then follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

(The partial sums of the last two displayed series differ by 1, so the convergence of one implies the convergence of the other.)

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n}}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ since

$$0 \leq \frac{1}{n^2 + \sqrt{n}} \leq \frac{1}{n^2}.$$

(c) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ since

$$0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

for all n, and the harmonic series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, diverges.

We will continue with our list of tests next time.