Math 112 lecture for Monday, Week 10

Series

(Supplemental reading: Section 9.1 in Swanson.)

Definition. Let $\{a_n\}$ be a sequence of real or complex numbers. The *n-th partial* sum of $\{a_n\}$ is

$$s_n := \sum_{i=1}^{n} a_i.$$

Example. The third partial sum of $\{1/n\}_{n\geq 1}$ is

$$s_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}.$$

Note in the following definition that *a series is a special kind of sequence*, and, hence, all our earlier results about sequences apply.

Definition. Let $\{a_n\}$ be a sequence of real or complex numbers. The *infinite series* whose n-th term is a_n is the sequence of partial sums

$$\{s_n\} = a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, \dots$$

If $\{s_n\}$ converges, say $\lim s_n = s$, then we write

$$\sum_{i=1}^{\infty} a_i := \lim_{n\to\infty} s_n = s,$$

and s is called the sum of the series. If $\{s_n\}$ diverges, we say the series diverges.

Example. Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{n(n+1)}.$$

The first few partial sums are

$$s_1 = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$s_2 = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$s_3 = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} = \frac{3}{4}.$$

It looks like

$$s_n = \frac{n-1}{n}. \tag{1}$$

If that's the case, then

$$\sum_{i=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

We could easily establish equation (1) by induction, but here is another (somewhat tricky) approach: note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus, the n-th partial sum for the series is

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} - \frac{1}{n+1}.$$

The above sum is called a *telescoping sum*—all its intermediate terms collapse. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} := \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1}\right) = 1.$$

Figure 1: A collapsible telescope.

Proposition. (Limit theorem for series.) Suppose that $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$, and let r a real or complex number. Then

(a) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = a + b.$

(b) $\sum_{n=1}^{\infty}(ra_n) = r\sum_{n=1}^{\infty} a_n = ra.$

Proof. Let s_n and t_n be the n-th partial sums for $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively. We are given that $\lim_{n\to\infty} s_n = a$ and $\lim_{n\to\infty} t_n = b$. From our limit theorems for ordinary limits, we have $\lim_{n\to\infty}(s_n + t_n) = a + b$ and $\lim_{n\to\infty}(rs_n) = ra$. But $s_n + t_n$ is the n-th partial sum for $\sum_{n=1}^{\infty}(a_n + b_n)$ and rs_n is the n-th partial sum for $\lim_{n=1}^{\infty}(ra_n) = ra$. The result follows. $\square$

GEOMETRIC SERIES

Definition. A *geometric series* is a series of the form

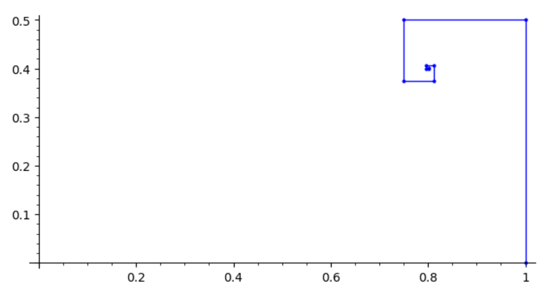
$$\sum_{n=0}^{\infty} r^n$$

where r is a real or complex number.

Example. Consider the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n$. The index starts at $n = 0$, so the first few terms of the sequence of partial sums is

$$s_0 = 1,\ s_1 = 1 + \frac{i}{2},\ s_2 = 1 + \frac{i}{2} - \frac{1}{4},\ s_3 = 1 + \frac{i}{2} - \frac{1}{4} - \frac{i}{8},\ s_4 = 1 + \frac{i}{2} - \frac{1}{4} - \frac{i}{8} + \frac{1}{16}.$$

Here is a picture of the sequence of partial sums (connected by lines)



Theorem. (Geometric series) Let $r \in \mathbb{C}$.

(a) If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

More generally if $|r| < 1$ and $m \in \mathbb{N}$, then

$$\sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r}.$$

(b) If $|r| \geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.

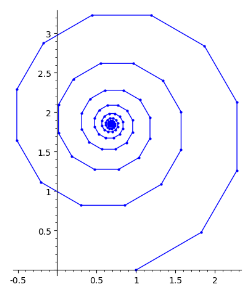
Example. Let

$$\alpha = (20/21)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{2.1} + \frac{\sqrt{3}}{2.1}i,$$

a point with argument 30° and length slightly less than 1. According to the geometric series theorem, the series converges to

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \approx 0.68 + 1.85\, i.$$

Here is a picture of the partial sums for $\sum_{n=0}^{\infty} \alpha^n$



Note that the angle at which each successive line rotates is $\arg(\alpha) = 30°$. Can you see why?

Example Compute the following sums:

(a) $\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$ (b) $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$ (c) $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$

(d) $\displaystyle\sum_{n=0}^{\infty} (4i)^n$ (e) $\displaystyle\sum_{n=3}^{\infty} 7\frac{2^{2n}}{10^n}$.

Note that in (c), the sum starts with $n = 2$, and in (e), the sum starts with $n = 3$.

SOLUTION: (a) Since $|i/2| = 1/2 < 1$, the formula for the sum of a geometric series applies, and we get

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2}{2-i} \cdot \frac{2+i}{2+i} = \frac{4+2i}{5}.$$

(b) Using the limit theorems for series and the formula for the sum of a geometric series, we have

$$\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n = 5\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 5 \cdot \frac{1}{1-\frac{2}{3}} = 15.$$

(c) Using the formula for a geometric series, we have

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = \frac{(3/4)^2}{1-\frac{3}{4}} = \frac{9}{4}.$$

Note: Another way to approach this problem is to first compute the sum starting with $n = 0$:

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{3}{4}} = 4,$$

and then subtract the terms corresponding to $n = 0, 1$:

$$4 - 1 - \frac{3}{4} = \frac{9}{4}.$$

This method takes more steps and, in practice, is a likely source of arithmetic errors.

(d) Since $|4i| = 4 \geq 1$, the series $\sum_{n=0}^{\infty} (4i)^n$ diverges.

(e) We have

$$\begin{aligned}\sum_{n=3}^{\infty} 7\frac{2^{2n}}{10^n} &= \sum_{n=3}^{\infty} 7\frac{4^n}{10^n} \\ &= 7\sum_{n=3}^{\infty} \left(\frac{4}{10}\right)^n \\ &= 7\sum_{n=3}^{\infty} \left(\frac{2}{5}\right)^n\end{aligned}$$

$$= 7\left(\frac{2}{5}\right)^3 \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$$

$$= 7\left(\frac{2}{5}\right)^3 \frac{1}{1-2/5}$$

$$= 7\left(\frac{2}{5}\right)^3 \frac{5}{3} = \frac{7 \cdot 8}{25 \cdot 3} = \frac{56}{75}.$$

Proof of the geometric series theorem. An easy induction argument shows that

$$\sum_{i=0}^{n} r^n = \frac{1-r^{n+1}}{1-r}$$

for $r \neq 1$. Suppose that $|r| < 1$. By the above formula, the n-th partial sum of the series is

$$s_n = \frac{1-r^{n+1}}{1-r}.$$

Using our limit theorems, we get

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} \\
&= \frac{1}{1-r} \cdot \lim_{n\to\infty} (1-r^{n+1}) && \left(\text{since } \frac{1}{1-r} \text{ is a constant}\right) \\
&= \frac{1}{1-r}\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} r^{n+1}\right) \\
&= \frac{1}{1-r}\left(\lim_{n\to\infty} 1 - 0\right) && (\text{since } |r| < 1) \\
&= \frac{1}{1-r}.
\end{aligned}$$

Next, use our limit theorem for series to get

$$\sum_{n=m}^{\infty} r^n = r^m + r^{m+1} + r^{m+2} + \cdots = r^m(1 + r + r^2 + \dots) = r^m \sum_{n=0}^{\infty} r^n = \frac{r^m}{1-r}.$$

Now suppose that $|r| \geq 1$. If $r = 1$, then the n-th partial sum of the series is $s_n = n$, which gives a divergent sequence. Next, suppose that $|r| \geq 1$ and $r \neq 1$. Recall that in that case, we showed earlier that $\lim_{n\to\infty} r^n$ diverges. For the sake of contradiction suppose the series converges. Say $\sum_{n=0}^{\infty} r^n = s$. From our earlier formula, we know

$$s_n = \frac{1 - r^{n+1}}{1 - r}.$$

Solve for r^{n+1} to get $r^{n+1} = 1 - (1 - r)s_n$, and hence,

$$r^n = \frac{1}{r}(1 - (1 - r)s_n).$$

We are assuming $\lim_{n\to\infty} s_n = s$, and therefore, using our limit theorems

$$\lim_{n\to\infty} r^n = \frac{1}{r}(1 - (1 - r)s).$$

However, we know $\lim_{n\to\infty} r^n$ diverges when $|r| \geq 1$ and $r \neq 1$. This contradiction shows that $\sum_{n=0}^{\infty} r^n$ must diverge. $\square$