Math 112 lecture for Friday, Week 10

Series tests II

(Supplemental reading: Section 9.2 in Swanson.)

We continue our discussion of the standard tests for determining whether a series converges.

1. the geometric series test
2. the n-th term test
3. the comparison test
4. **the limit comparison test**
5. **the alternating series test**
6. **the absolute convergence test**
7. the ratio test
8. the root test
9. the integral test
10. the p-series test.

4. The limit comparison test. In the following, note the key hypotheses that the sequences have *positive* terms and that the limit of their quotient is *nonzero*. Since a sequence converges if and only if its tail converges, one may also apply the limit comparison test to sequences that are positive after a finite number of terms.

Proposition (limit comparison test). Suppose $\{a_n\}$ and $\{b_n\}$ are real sequences of *positive* terms and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0.$$

Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Remark. Whether $\sum_n a_n$ converges depends upon how quickly the terms a_n die off. For instance, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not. Even though $\lim \frac{1}{n^2} = \lim \frac{1}{n} = 0$, the terms of $\left\{\frac{1}{n}\right\}$ do not die off quickly enough. The condition $\frac{a_n}{b_n} = L \neq 0$ means that a_n and b_n grow or diminish at comparable rates, unlike the case of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$.

Example. Consider the series

$$\sum_{n=1}^{\infty} \frac{6n+7}{2n^2-4}.$$

For large n,

$$\frac{6n+7}{2n^2-4} \approx \frac{6n}{2n^2} = \frac{3}{n}.$$

Since the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we expect $\sum_{n=1}^{\infty} \frac{6n+7}{2n^2-4}$ will also diverge. *To make this intuition precise*, we use the limit comparison test with $a_n = \frac{6n+7}{2n^2-4}$ and $b_n = \frac{1}{n}$:

$$\frac{a_n}{b_n} = \frac{6n+7}{2n^2-4} \cdot \frac{n}{1} = \frac{6n^2+7n}{2n^2-4} \longrightarrow 3 \neq 0$$

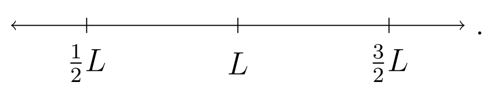
as $n \to \infty$. Thus, $\sum_{n=1}^{\infty} \frac{6n+7}{2n^2-4}$ diverges by limit comparison with the harmonic series.

Exercise. By a similar argument, show that $\sum_{n=1}^{\infty} \frac{7}{2n^2-4}$ converges using limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Proof of the limit comparison test. Suppose that $\lim \frac{a_n}{b_n} = L \neq 0$. Since $a_n > 0$, $b_n > 0$, and $L \neq 0$, we have $L > 0$. Apply the definition of the limit with $\varepsilon = \frac{1}{2}L > 0$ to find N such that $n > N$ implies

$$\left| L - \frac{a_n}{b_n} \right| < \frac{1}{2}L.$$

Thus, $\frac{a_n}{b_n}$ is within a distance of $L/2$ from L, i.e., it is in the interval pictured below:



Then

$$\frac{1}{2}L < \frac{a_n}{b_n} < \frac{3}{2}L \quad \Rightarrow \quad \frac{1}{2}Lb_n < a_n < \frac{3}{2}Lb_n.$$

If $\sum a_n$ converges, the ordinary comparison theorem implies that $\sum \frac{1}{2}Lb_n$ converges. Hence, using our earlier limit theorems[1], so does $\frac{2}{L}\sum \frac{1}{2}Lb_n = \sum b_n$. Similarly, if $\sum b_n$ converges, then so does $\frac{3}{2}L\sum b_n = \sum \frac{3}{2}Lb_n$. Then since $a_n < \frac{3}{2}Lb_n$, the comparison theorem says $\sum a_n$ converges. $\square$

[1] If $\sum c_n$ converges and k is a constant, then $\sum kc_n$ converges and equals $k\sum c_n$.

5. The alternating series test.

We saw earlier that if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. We also saw that the converse does not hold, in general. For instance $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$. We now discuss the case of a special type of series for which the converse does hold.

Proposition (alternating series test). Let $\{a_n\}$ be a monotonically decreasing sequence of positive terms. Then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

Examples. The *alternating harmonic series* is the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

To see that it converges, apply the alternating series test with $a_n = \frac{1}{n}$. (Check that the test applies: $\left\{\frac{1}{n}\right\}$ is a *monotonically decreasing* sequence of *positive* terms, and $\lim_{n\to\infty} \frac{1}{n} = 0$.)

The alternating harmonic series is the canonical example of what is called a *conditionally convergent* series: it converges, but the series formed by taking the absolute values of it terms, $\sum_{n=1}^{\infty} |(-1)^{n+1}\frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$, diverges.

Proof of the alternating series test.

($\Rightarrow$) Suppose that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. By the n-th term test, it follows that $\lim_{n\to\infty}(-1)^{n+1}a_n = 0$. We have seen that if $\{b_n\}$ is any sequence, real or complex, that $\lim_{n\to\infty} b_n = 0$ if and only if $\lim_{n\to\infty} |b_n| = 0$. Applying that here, we conclude that $\lim_{n\to\infty} a_n = 0$.

($\Leftarrow$) Now suppose that $\lim_{n\to\infty} a_n = 0$. We must argue that the sequence of partial sums $\{s_n\}$ for $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Our strategy is to divide these partial sums into two subsequences, $\{s_{2n}\}$ and $\{s_{2n+1}\}$—the even- and the odd-indexed terms—and to argue that these two subsequences converge to the same value. First consider $\{s_{2n}\}$:

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}).$$

The terms of s_{2n} are grouped as above to make it clear that since $\{a_n\}$ is monotonically decreasing, each of the $a_{2k-1} - a_{2k}$ is nonnegative. Thus, $\{s_{2n}\}$ is monotonically increasing. Further, looking at this sequence a different way makes it clear that s_{2n} is bounded above by a_1:

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1.$$

By the monotone convergence theorem, we conclude that

$$\lim_{n\to\infty} s_{2n} = s$$

for some $s \in \mathbb{R}$. It then follows that

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + a_{n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{n+1} = s + 0 = s.$$

(We know that $\lim_{n\to\infty} a_{n+1} = 0$ by the n-term test.) Since

$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = s,$$

it looks like there is some hope of showing that $\lim_{n\to\infty} s_n = s$. In fact, that is true, and we can give a straight ε-N proof. Let $\varepsilon > 0$, and then take N so that $n > N$ implies.

$$\lim_{n\to\infty} |s - s_{2n}| < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} |s - s_{2n+1}| < \varepsilon$$

simultaneously. But this just says that every term of the sequence $\{s_n\}$ is within ε of s: $n > 2N + 1$ implies $|s - s_n| < \varepsilon$. □

6. The absolute convergence test.

Definition. A series of complex numbers $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent. If $\sum a_n$ converges but $\sum |a_n|$ does not, then $\sum a_n$ is *conditionally convergent.*

Example. To emphasize the example presented earlier: the alternating harmonic series, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$, converges (by the alternating series test), but $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ does not. So the alternating harmonic series is conditionally convergent.

Many of our series tests apply to only series whose terms are nonnegative reals. The following proposition is of central importance: it shows how these series tests can say something about arbitrary complex series (since the absolute value of a complex number is nonnegative and real).

Proposition (absolute convergence test). Let $\sum a_n$ be a complex series. Then if $\sum a_n$ is absolutely convergent, it is convergent:

$$\sum |a_n| \text{ convergent} \quad \Longrightarrow \quad \sum a_n \text{ convergent.}$$

Examples. The series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

converges by the absolute convergence test since

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

and we have seen that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Similarly,

$$\sum_{n=1}^{\infty}\frac{\cos(n)+i\sin(n)}{n^2}$$

is absolutely convergent since

$$\sum_{n=1}^{\infty}\left|\frac{\cos(n)+i\sin(n)}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos(n)+i\sin(n)|}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Proof of the absolute convergence test. Suppose that $\sum_{n=1}^{\infty}|a_n|$ converges. We will use the fact that a real or complex series converges if and only it is a Cauchy sequence in order to prove that $\sum_{n=1}^{\infty}a_n$ converges.

Define

$$s_n = a_1 + \cdot + a_n \quad \text{and} \quad \tilde{s}_n = |a_1| + \cdot + |a_n|,$$

the partial sums for $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}|a_n|$, respectively. We are given that $\{\tilde{s}_n\}$ converges and must show that $\{s_n\}$ converges. Let $\varepsilon > 0$. Since $\{\tilde{s}_n\}$ converges, it is a Cauchy sequence. Thus, there exists N such that $m, n > N$ implies

$$|\tilde{s}_m - \tilde{s}_n| < \varepsilon.$$

Without loss of generality, suppose that $m \geq n$. Then, using the triangle inequality,

$$\begin{aligned}
|s_m - s_n| &= |a_{n+1} + \cdots + a_m| \\
&\leq |a_{n+1}| + \cdots + |a_m| \\
&= ||a_{n+1}| + \cdots + |a_m|| \\
&= |\tilde{s}_m - \tilde{s}_n| \\
&< \varepsilon.
\end{aligned}$$

Thus, $\{s_n\}$ is a Cauchy sequence and therefore converges. In other words, $\sum_{n=1}^{\infty}a_n$ converges. □

A peculiar property of conditionally convergent series. What happens if you rearrange the terms of an infinite series? To be precise, define a *rearrangement* of a series $\sum a_n$ to be a series $\sum b_n$ where the elements of $\{a_n\}$ and $\{a_n\}$ are in bijection—same sums but in different orders. By the commutative law for addition, one might expect to get the same behavior: one of the series converges if and only if the other does, and it they do converge, they converge to the same value. It turns out that *this is true for absolutely convergent series but not for conditionally convergent ones.* For instance, all rearrangements of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ will converge to the same value. On the other hand the value of a rearrangement of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ will depend on the rearrangement—even whether the series will converge. Something even wilder is true:

Proposition. Let $\sum a_n$ be a conditionally convergent real series, and let a be any real number. Then there is a rearrangement of $\sum a_n$ that converges to a. Further, there are rearrangements of $\sum a_n$ that diverge to ∞, that diverge to $-\infty$, and that fail to have any limit.

The idea behind the proof is the following. Say $\sum a_n$ is conditionally convergent. Then let $\sum p_n$ be the same as $\sum a_n$ after setting all negative terms, $a_n < 0$, equal to zero. Similarly, let $\sum q_n$ be the same as $\sum a_n$ after setting all positive terms equal to zero. Using the monotone convergence theorem, one may show that since $\sum a_n$ is conditionally convergent, $\sum p_n$ diverges to ∞ and $\sum q_n$ diverges to $-\infty$. To get a rearrangement of $\sum a_n$ that converges to an arbitrary real number a do the following. Assume $a > 0$, the case of $a \leq 0$ being similar. Create the rearrangement of $\sum a_n$ in steps. First add enough terms of $\sum p_n$ until we first get a number bigger than a. That's possible since $\sum p_n$ diverges to ∞. Next, add enough terms from $\sum q_n$ until we first get a number less than a, possible since $\sum q_n$ diverges to $-\infty$. Continue now by adding further terms from $\sum p_n$ until we first get a number above a, and then add further terms from $\sum q_n$ until we first get below a. Continue ad infinitum. The next part of the argument is to show the resulting rearrangement converges to a. A similar argument holds in the cases where $a = \pm\infty$.