Math 112 lecture for Wednesday, Week 9

Monotone Convergence Theorem

(Supplemental reading: Section 8.6 Swanson.)

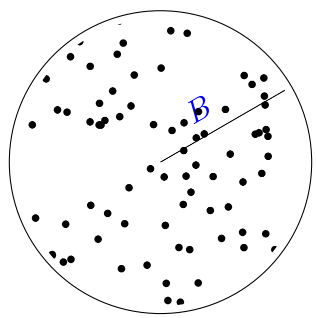
Today, we will prove two results: (i) convergent sequences are bounded (but not conversely), and (ii) the Monotone Convergence Theorem.

Definition. A sequence $\{s_n\}$ of complex numbers is *bounded* if there exists $B \in \mathbb{R}$ such that

$$|s_n| \leq B$$

for all n.

If the s_n are all real numbers, then this definition of boundedness coincides with the one we gave earlier. In general, it means that all of the numbers in the sequence are containing in a closed ball of finite radius centered at the origin:



Theorem. Every convergent sequence is bounded.

Proof. Let $\{s_n\}$ be a convergent sequence, and say $\lim_{n\to\infty} s_n = s$. Apply the definition of the limit with $\varepsilon = 1$ to find an $N \in \mathbb{R}$ such that $n > N$ implies

$$|s - s_n| < 1.$$

By taking N larger, if necessary, we may assume N is an integer greater than or equal to 1. This will be usual later on in the proof. We apply the reverse triangle inequality:

$$1 > |s - s_n| = |s_n - s| \geq |s_n| - |s| \quad \Rightarrow \quad 1 + |s| > |s_n|.$$

Thus, if $n > N$ then $|s_n| < 1 + |s|$. So we have accomplished the hardest part by bounding all but finitely many points in the sequence by the real number $1 + |s|$, but it could be that $s_1, s_2, \dots, s_N$ are further away from the origin. So let

$$B = \max \{|s_1|, |s_2|, \dots, |s_N|, 1 + |s|\}.$$

This means that we have set B equal to the maximum number in the given set. It follows that

$$|s_n| \le B$$

for all n: If $n > N$, then $|s_n| < 1 + |s| \le B$, and if $n \le N$, then $|s_n| \le |s_n| \le B$. □

Remark. The converse of the above theorem does not hold. For instance, consider the sequence $1, -1, 1, -1, \dots$ It is bounded but does not converge.

MONOTONE CONVERGENCE THEOREM

Definition. A sequence of real numbers $\{s_n\}$ is *monotone increasing* if $s_n \le s_{n+1}$ for all n. It is *monotone decreasing* if $s_n \ge s_{n+1}$ for all n.

Note that *monotone increasing* is synonymous with *non-decreasing* since the condition is $s_n \le s_{n+1}$, not $s_n < s_{n+1}$. An analogous remark holds for monotone decreasing sequences.

Example. The constant sequence $1, 1, 1, \dots$ is both monotone increasing and monotone decreasing.

Theorem. (Monotone Convergence Theorem, MCT). If $\{s_n\}$ is a monotone increasing sequence that is bounded above or a monotone decreasing sequence that is bounded below, then $\{s_n\}$ converges. (And it converges to $\sup \{s_n\}$ or $\inf \{s_n\}$, respectively.)

Proof. Suppose that $\{s_n\}$ is monotone increasing and bounded above. (The case where $\{s_n\}$ is monotone decreasing and bounded below is similar, or it can be reduced to the increasing case by considering $\{-s_n\}$.) By completeness of the real numbers, the *set* $\{s_n\}$ has a supremum. Say $s = \sup \{s_n\}$. We claim that $\lim_{n\to\infty} s_n = s$. To see this, let $\varepsilon > 0$. Then $s - \varepsilon < s$, i.e., $s - \varepsilon$ is strictly less than the least upper bound of $\{s_n\}$. This means that $s - \varepsilon$ is not an upper bound for that set. Therefore, there exists some s_N such that $s - \varepsilon < s_N$, i.e., $s - s_N < \varepsilon$. However, since the sequence $\{s_n\}$ is a monotone increasing sequence, if $n > N$, it follows that $s_n \ge s_N$, and hence, $-s_n \le -s_N$. Adding s to both sides yields $s - s_n \le s - s_N$. Now, since s is an upper bound for the set, we also have $0 \le s - s_n$. Putting this all together:

$$0 \le s - s_n \le s - s_N < \varepsilon$$

for all $n > N$. It follows that $|s - s_n| < \varepsilon$ for all $n > N$. We have shown that $\lim_{n\to\infty} s_n = s$, as claimed. $\square$

Example. Here is an example of the monotone convergence theorem in use. Define $s_1 = 1$, and for each $n \geq 1$, define $s_{n+1} := \sqrt{1 + s_n}$. The first few terms of the sequence are

$$1,\ \sqrt{1+1},\ \sqrt{1+\sqrt{1+1}}, \ldots$$

To find the limit, we first appeal to the monotone convergence theorem. We show the sequence is monotone increasing, i.e., that $s_n \leq s_{n+1}$ by induction. For the base case, $n = 1$, note that $s_1 = 1 \leq s_2 = \sqrt{2}$. Now suppose that $s_n \leq s_{n+1}$ for some $n \geq 1$. It follows that

$$s_{n+1} = \sqrt{1 + s_n} \leq \sqrt{1 + s_{n+1}} = s_{n+2}.$$

(Here, we have used that $f(x) = \sqrt{x}$ is an increasing function: if $x < y$, then $\sqrt{x} < \sqrt{y}$.) The result now follows for all n by induction.

To use the monotone convergence theorem, we must also verify that the sequence is bounded above. We show it's bounded above by 2 by induction. For the base case, note that $s_1 = 1 \leq 2$. Suppose that $s_n \leq 2$ for some $n \geq 1$. Then

$$s_{n+1} = \sqrt{1 + s_n} \leq \sqrt{1 + 2} = \sqrt{3} \leq 2.$$

The result holds for all n by induction.

The monotone convergence theorem now tells us the sequence has a limit. Say that $\lim_{n\to\infty} s_n = s$. We would like to evaluate s. We will appeal to a result we have not yet shown: if $f(x)$ is a continuous function, then $\lim_{n\to\infty} f(s_n) = f(\lim_{n\to\infty} s_n) = f(s)$. We apply this in the case $f(x) = \sqrt{x}$:

$$s_{n+1} = \sqrt{1 + s_n} \quad \Rightarrow \quad \lim_{n\to\infty} s_{n+1} = \sqrt{\lim_{n\to\infty} (1 + s_n)} = \sqrt{1 + s}.$$

It is an easy exercise to show that $\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} s_n = s$.[1] Therefore, we see

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1} = \sqrt{1 + s}.$$

Squaring both sides,

$$s^2 = 1 + s \quad \Rightarrow \quad s^2 - s - 1 = 0 \quad \Rightarrow \quad s = \frac{1 \pm \sqrt{5}}{2}$$

[1] Here we are comparing the sequence $\{s_n\}$ to the "shifted sequence" $\{s_{n+1}\} = s_2, s_3, \ldots$. If s_n is within ε of s for all $n > N$, then so is s_{n+1} since $n > N$ implies that $n + 1 > N$.

by the quadratic equation. By squaring, we introduced an extraneous answer. Since $0 \leq s_n$ for all n, it follows that $0 = \lim 0 \leq \lim s_n = s$.[2] Hence, s must be nonnegative. It follows that

$$s = \frac{1+\sqrt{5}}{2}.$$

[2] In general, taking limits "preserves inequalities". We will discuss this later and assume it for now.