Math 112 lecture for Monday, Week 9

New-from-old limit theorem

(Supplemental reading: Theorem 8.4.3 in Swanson.)

In general, it is difficult to give limit proofs using ε-N arguments. We give an example of this difficulty, and then develop a technique for a much easier, high-level approach.

Example. $\displaystyle\lim_{n\to\infty} \frac{n^3+2n}{5n^3+2} = \frac{1}{5}$.

Proof. Given $\varepsilon > 0$, let $N = 2/\varepsilon$, and suppose $n > N$. Then

$$\begin{aligned}
\left|\frac{n^3+2n}{5n^3+2} - \frac{1}{5}\right| &= \left|\frac{5(n^3+2n)-(5n^3+2)}{5(5n^3+2)}\right| \\
&= \left|\frac{10n-2}{5(5n^3+2)}\right| \\
&< \left|\frac{10n-2}{5n^3+2}\right| \\
&< \left|\frac{10n-2}{5n^3}\right| \\
&< \left|\frac{10n}{5n^3}\right| \\
&= \left|\frac{2}{n^2}\right| \\
&< \left|\frac{2}{n}\right| \\
&= \frac{2}{n} < \frac{2}{N} \\
&= \frac{2}{2/\varepsilon} \\
&= \varepsilon.
\end{aligned}$$

□

Limit theorems. Often, a sequence can constructed from simpler sequences using algebraic operations (addition, substraction, multiplication, and division). In this case, the following theorem is useful:

Theorem. (New-from-old limit theorem.) Suppose that $\lim_{n\to\infty} s_n = s$ and that $\lim_{n\to\infty} t_n = t$. Then,

(1) $\lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t$.

(2) $\lim_{n\to\infty}(s_n t_n) = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n) = st$.

(3) $\displaystyle\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} = \frac{s}{t}$ provided $t \neq 0$.

Corollary. If a and b are real or complex constants, then $\lim_{n\to\infty}(as_n) = a\lim_{n\to\infty} s_n = as$ and, more generally, $\lim_{n\to\infty}(as_n + bt_n) = a\lim_{n\to\infty} s_n + b\lim_{n\to\infty} t_n = as + bt$.

We will provide proofs of these results at the end of these notes. (**Note:** Pay particular attention to the proof of part (1) given there. It contains some important, generally useful ideas.) First, though, we will illustrate the use of our theorem.

Building blocks. We have seen that for $c \in \mathbb{C}$, the constant sequence has limit c, i.e., $\lim_{n\to\infty} c = c$. And we have also seen that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

In the following examples, we will see how complicated sequences can be evaluated using these building blocks and the new-from-old limit theorem (LT). Note the important trick in part (d).

Examples.

(a)

$$\lim_{n\to\infty}\left(5 + \frac{1}{n}\right) = \lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{1}{n} \qquad \text{(LT 1)}$$

$$= 5 + 0 = 5.$$

(b)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^2} &= \lim_{n\to\infty} \left(\frac{1}{n}\cdot\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) && \text{(LT 2)} \\
&= 0\cdot 0 = 0.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{5}{n^2} &= \lim_{n\to\infty} \left(5\cdot\frac{1}{n}\cdot\frac{1}{n}\right) \\
&= (\lim_{n\to\infty} 5)\left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) && \text{(LT 2)} \\
&= 5\cdot 0 = 0.
\end{aligned}
$$

(d)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n^3+2n}{5n^3+2} &= \lim_{n\to\infty} \frac{\frac{1}{n^3}(n^3+2n)}{\frac{1}{n^3}(5n^3+2)} && \text{(NB: important trick)} \\
&= \lim_{n\to\infty} \frac{1+\frac{2}{n^2}}{5+\frac{2}{n^3}} \\
&= \frac{\lim_{n\to\infty}\left(1+\frac{2}{n^2}\right)}{\lim_{n\to\infty}\left(5+\frac{2}{n^3}\right)} && \text{(LT3)} \\
&= \frac{\lim_{n\to\infty} 1+\lim_{n\to\infty}\frac{2}{n^2}}{\lim_{n\to\infty} 5+\lim_{n\to\infty}\frac{2}{n^3}} && \text{(LT1)} \\
&= \frac{1+\lim_{n\to\infty}\frac{2}{n^2}}{5+\lim_{n\to\infty}\frac{2}{n^3}} && \text{(constant sequences)} \\
&= \frac{1+0}{5+0} && \text{(as in example 3)} \\
&= \frac{1}{5}.
\end{aligned}
$$

Proof of new-from-old limit theorem.

For part (1), we need to show that

$$\lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t.$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, there exists N_s such that $n > N_s$ implies

$$|s - s_n| < \frac{\varepsilon}{2}.$$

Similarly, since $\lim_{n\to\infty} t_n = t$, there exists N_t such that $n > N_t$ implies

$$|t - t_n| < \frac{\varepsilon}{2}.$$

Define N to be the maximum of N_s and N_t:

$$N := \max\{N_s, N_t\}.$$

Thus, $N \geq N_s$ and $N \geq N_t$.

Suppose $n > N$. Then, since $n > N \geq N_s$, it follows that

$$|s - s_n| < \frac{\varepsilon}{2}.$$

Similarly, since $n > N \geq N_t$, it follows that

$$|t - t_n| < \frac{\varepsilon}{2}.$$

Hence, if $n > N$,

$$|(s+t) - (s_n + t_n)| = |(s - s_n) + (t - t_n)| \leq |s - s_n| + |t - t_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That completes the proof of part (1).

For part (2), we show that

$$\lim_{n\to\infty}(s_n t_n) = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n) = st.$$

Given $\varepsilon > 0$, define

$$\eta = \min\left\{1, \frac{\varepsilon}{1 + |s| + |t|}\right\}.$$

This means η is the minimum of the two displayed quantities. So $\eta \leq 1$ and $\eta \leq \varepsilon/(1 + |s| + |t|)$ (and equality holds in at least one of these). As above, since $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, we can find a single N such that $n > N$ implies

$$|s - s_n| < \eta \quad \text{and} \quad |t - t_n| < \eta. \tag{1}$$

Now for the creative part: we can write

$$s_n t_n - st = (s - s_n)(t - t_n) - s(t - t_n) - t(s - s_n).$$

(Check this by multiplying out the expression on the right.) It follows that

$$\begin{aligned}
|st - s_n t_n| &= |s_n t_n - st| \\
&= |(s - s_n)(t - t_n) - s(t - t_n) - t(s - s_n)| \\
&\leq |(s - s_n)(t - t_n)| + |s(t - t_n)| + |t(s - s_n)| && \text{(triangle inequality)} \\
&= |(s - s_n)||(t - t_n)| + |s||(t - t_n)| + |t||(s - s_n)| \\
&< \eta \cdot \eta + |s|\eta + |t|\eta && \text{(Equation (1))} \\
&\leq \eta + |s||\eta + |t|\eta && \text{(since } \eta \leq 1\text{)} \\
&= \eta(1 + |s| + |t|) \\
&< \varepsilon && \left(\text{since } \eta \leq \frac{\varepsilon}{1 + |s| + |t|}\right).
\end{aligned}$$

The proof for part (3) is similar, and we leave it as an exercise for the interested reader. □

Proof of the corollary to the new-from-old limit theorem.
Suppose a and b are constants. Then

$$\begin{aligned}
\lim_{n\to\infty}(as_n + bt_n) &= \lim_{n\to\infty}(as_n) + \lim_{n\to\infty}(bt_n) && \text{(LT1)} \\
&= \lim_{n\to\infty} a \lim_{n\to\infty} s_n + \lim_{n\to\infty} b \lim_{n\to\infty} t_n && \text{(LT2)} \\
&= a \lim_{n\to\infty} s_n + b \lim_{n\to\infty} t_n && \text{(limit of constant seq.)} \\
&= as + bt.
\end{aligned}$$

To see that $\lim_{n\to\infty}(as_n) = a \lim_{n\to\infty} s_n$, let $b = 0$, above. □