Math 112 lecture for Friday, Week 9

Misc. limit theorems, infinite limits, and Cauchy sequences

(Supplemental reading: Sections 8.3, 8.4, 8.7, and 8.8 in Swanson.)

Miscellaneous limit theorems

We state several theorems here about limits. The numbering refers to Swanson's text, where the interested reader may find the proofs.

Theorem 8.4.1. Limits are unique: if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = s'$, then $s = s'$.

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = s'$, there exists an N such that $n > N$ implies that $|s - s_n| < \varepsilon/2$ and $|s' - s_n| < \varepsilon/2$. As usual, we can take N large enough so that it applies to both limits simultaneously. By the triangle inequality, we then have that for $n > N$,

$$|s - s'| = |(s - s_n) - (s' - s_n)| \leq |s - s_n| + |s' - s_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We have shown that $|s - s'| < \varepsilon$ for all $\varepsilon > 0$. Since there are no infinitely small positive numbers (proved in the earlier *Extrema* lecture), it follows that $|s - s'| = 0$, and hence, $s = s'$. □

Theorem 8.4.11. (Squeeze theorem.) Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be *real* sequences, and suppose that $a_n \leq b_n \leq c_n$ for all n. Then, if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Example. Here, we prove that $\lim\limits_{n\to\infty} \dfrac{n^2 - 3}{n^3 + 6n + 1} = 0$ using the squeeze theorem. We have

$$\frac{n^2 - 3}{n^3 + 6n + 1} \leq \frac{n^2}{n^3 + 6n + 1} \leq \frac{n^2}{n^3} = \frac{1}{n};$$

Thus,

$$0 \leq \frac{n^2 - 3}{n^3 + 6n + 1} \leq \frac{1}{n},$$

for $n > 1$. Given that $\lim_{n\to\infty} 0 = 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, it follows that

$$\lim_{n\to\infty} \frac{n^2 - 3}{n^3 + 6n + 1} = 0$$

by the squeeze theorem.

Example. Consider the sequence $\left\{\dfrac{\sin(n)}{n}\right\}$. Since $|\sin(n)| \leq 1$, we have

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}.$$

Then, since

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows that

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

by the squeeze theorem.

Theorem 8.4.10. The operation of taking limits preserves inequalities: Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent *real* sequences and that $a_n \leq b_n$ for all n (or for all n past a certain point) then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

(Recall that $\mathbb{C}$ cannot be ordered. So to above theorem does not make sense for complex sequences in general.)

Remark. The operation of taking limits does not preserve strict inequality. For example, compare the constant sequence $\{0\}$ with the sequence $\left\{\frac{1}{n}\right\}$. We have

$$0 < \frac{1}{n}$$

for all n, but

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n}.$$

Subsequences

We form a subsequence from a giving sequence by dropping terms from the sequence (but leaving an infinite number):

$$\begin{array}{rl} \text{sequence:} & s_1, \underline{s_2}, \underline{s_3}, \underline{s_4}, s_5, \underline{s_6}, s_7, s_8, \underline{s_9}, s_{10}, \underline{s_{11}}, \ldots \\ \text{subsequence:} & s_2, s_3, s_4, s_6, s_9, s_{11}, \ldots \end{array}$$

Definition. Let $\{s_n\}$ be a sequence, and let $n_0 < n_1 < n_2 < \dots$ be any sequence of natural numbers. Then the sequence $\{s_{n_k}\}_{k=0}^{\infty}$ is called a *subsequence* of $\{s_n\}$.

Example. In the previous example,

$$n_0 = 2, n_1 = 3, n_2 = 4, n_3 = 6, n_4 = 9, n_5 = 11, \dots$$

Example. Let $s_n = 1/n$ for $n = 1, 2, \dots$. Then the following is a subsequence of $\{s_n\}$:

$$\begin{aligned}\{s_{2k}\}_{k=1}^{\infty} &= s_2, s_4, s_6, s_8, \dots \\ &= \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\end{aligned}$$

Theorem. (Main theorem for subsequences.) If $\{s_n\}$ is a sequence converging to s and $\{s_{n_k}\}$ is any subsequence, then $\{s_{n_k}\}$ also converges to s. (Every subsequence of a convergent sequence is convergent and has the same limit.)

Example. The previous theorem is especially useful for proving non-convergence. For instance, consider the sequence $\{(-1)^n\}$. It has the constant sequence $\{1\}$ and the constant sequence $\{-1\}$ as subsequences. The former converges to 1 and the latter to -1. We can deduce that $\{(-1)^n\}$ diverges (since otherwise all subsequence would need to converge to the same value).

Infinite limits

Definition. A *real* sequence $\{a_n\}$ *diverges to* ∞ if for all $B \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that $n > N$ implies

$$a_n > B.$$

The sequence *diverges to* $-\infty$ if for all $B \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that $n > N$ implies

$$a_n < B.$$

We write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty,$$

respectively.

What this means: If $\lim_{n\to\infty} s_n = \infty$, and you come up with a number B, no matter how large, eventually, all of the terms in the sequences are larger than B. If $\lim_{n\to\infty} s_n = -\infty$, and you come up with a number B, no matter how negative, eventually, all of the terms in the sequences are less than B.

Example. To prove $\lim_{n\to\infty} n^2 = \infty$, let $B \in \mathbb{R}$. Given B, let $N = \sqrt{|B|}$. Then

$$n > N \quad \Rightarrow \quad n > \sqrt{|B|} \quad \Rightarrow \quad n^2 > |B| \quad \Rightarrow \quad n^2 > B.$$

Cauchy sequences.

The real numbers fill the "holes" in $\mathbb{Q}$. For instance, consider the set of rational numbers

$$S = \left\{x \in \mathbb{Q} : x^2 \le 2\right\}.$$

This is a nonempty set of $\mathbb{Q}$ and bounded above, but it does not have a supremum in $\mathbb{Q}$. One way to construct $\mathbb{R}$ is to use decimals. However that approach is a little tougher than it might first seem. (To point out one difficulty, two different decimals can be equal, e.g., $0.999\cdots = 1.000\ldots$.) We will now outline a different approach. There are sequences of rationals that "want to converge". For example, take the sequence we get by truncating the decimal expansion of $\sqrt{2}$:

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

However, without real numbers, this sequence has nowhere to converge to.

A key idea in one approach to the construction of $\mathbb{R}$ is to *think of sequences of rational numbers* as real numbers. For instance, we can think of the above sequence as $\sqrt{2}$. A real number that is already a rational number, e.g., $1/2$, can be thought of as the constant sequence $1/2, 1/2, 1/2, \ldots$ There are some obvious problems with this approach:

(a) We only want to consider sequences of rationals that "want to converge". What could this mean?

(b) There are many different sequences that want to converge to the same point. For instance, the constant sequence $\{0\}$ and the sequence $\{1/n\}$ both converge to 0, as do infinitely many other sequences.

We fix the first problem with the following definition:

Definition. A sequence of numbers (rational, real, or complex) $\{s_n\}$ is a *Cauchy sequence* if for $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $m, n > N$ implies

$$|s_m - s_n| < \varepsilon.$$

Remark. The rough idea behind this definition is that as you go out far in the sequence, the points in the sequence start to clump together—the distance between all the remaining points is small.

Proposition. Every convergent sequence is a Cauchy sequence.

Proof. Suppose that $\{s_n\}$ is a convergent sequence, and let $\varepsilon > 0$. Say $\lim_{n\to\infty} s_n = s$. Then there exists N such that $n > N$ implies

$$|s - s_n| < \frac{\varepsilon}{2}.$$

But then, if $m, n > N$, by the triangle inequality,

$$|s_m - s_n| = |(s - s_n) - (s - s_m)| \leq |s - s_n| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We've shown that $|s_m - s_n| < \varepsilon$ for all $m, n > N$. Hence $\{s_n\}$ is Cauchy. □

Theorem. A sequence of real or complex numbers is convergent if and only if it is a Cauchy sequence.

Proof. See Swanson's text, Section 8.7. □

Remark. The above theorem does not hold for $\mathbb{Q}$. For instance, take a sequence of rational numbers that converges to $\sqrt{2}$ in $\mathbb{R}$. That sequence will be a Cauchy sequence of rational numbers, but it will not converge in the rational numbers.

Definition of the real numbers

Let $\mathcal{C}$ be the set of all Cauchy sequences of rational numbers. As a first approximation, we could try to define $\mathbb{R} := \mathcal{C}$. The problem with this is that multiple Cauchy sequences will want to have the same limit. Again, for instance, consider the constant sequence 0 and the sequence $\{1/n\}$. To fix that we define an equivalence relation:

Definition. Let $\{s_n\}, \{t_n\} \in \mathcal{C}$. We say

$$\{s_n\} \sim \{t_n\}$$

if

$$\lim_{n\to\infty} (s_n - t_n) = 0.$$

Any easy check shows that $\sim$ is an equivalence relation on $\mathcal{C}$. An equivalence class for the relation is the set of all Cauchy sequences of rational numbers that want to converge to the same thing. We can then define the real numbers to be the sets of these equivalence classes:

Definition. The *real numbers* are

$$\mathbb{R} := \mathcal{C}/\!\sim .$$

with addition and multiplication defined by

$$[\{s_n\}] + [\{t_n\}] := [\{s_n + t_n\}]$$

$$[\{s_n\}][\{t_n\}] := [\{s_n t_n\}].$$

There are a lot of details to check:

(a) Are addition and multiplication well-define or do they depend on the choice of representatives for the equivalence classes? (Recall we had the same consideration when defining addition and multiplication for equivalence classes of integers modulo n.)

(b) Do we get a field?

(c) How do we define an order relation on $\mathbb{R}$?

We'll stop here, though.