Math 112 lecture for Wednesday, Week 7

Sequences I

(Supplemental reading: Start reading Chapter 8 in Swanson.)

A *sequence of complex numbers* (a *complex sequence*) is a function

$$\mathbb{N}_{>0} \to \mathbb{C}.$$

If s is such a function, instead of $s(n)$, we usually write s_n. Essentially, the function s is just an unending ordered sequence of numbers:

$$s_1, s_2, s_3, \ldots$$

The sequence s can be notated in various ways, including

$$\{s_n\}_{n>0}\,,\ \{s_n\}_{n=1}^{\infty}\,,\ \{s_n\}_n\,,\ \{s_n\}\,,$$

among others. A *real sequence* is a special case of a complex sequence in which the image of s is in $\mathbb{R} \subset \mathbb{C}$.

Example. Some examples of sequences:

(a)

$$\{1\}_{n=1}^{\infty} = 1, 1, 1, \ldots$$

(b) The *first term* in the following sequence is 4:

$$\{n\}_{n\geq 4} = 4, 5, 6, \ldots$$

(c)

$$\{(-1)^n\}_{n\geq 0} = 1, -1, 1, -1, \ldots$$

(d)

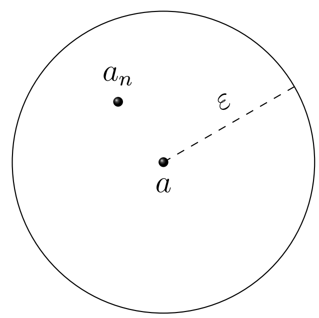
$$\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty} = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$$

Definition. Let $\{a_n\}$ be a sequence of complex numbers, and let $a \in \mathbb{C}$. Then the *limit of the sequence* $\{a_n\}$ *as* n *goes to infinity is* a, denoted $\lim_{n\to\infty} a_n = a$, if for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $n > N$ implies $|a - a_n| < \varepsilon$. If $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{C}$, we say $\{a_n\}$ is *convergent*, and if there is no such a, we say the sequence *diverges*.

Here is a shorthand for writing the definition: $\lim_{n\to\infty} a_n = a$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ such that $n > N \Rightarrow |a - a_n| < \varepsilon$. The symbols $\forall$ and $\exists$ are called *quantifiers*. They stand for "for all" and "there exists", respectively.

It is notoriously difficult to fully appreciate all that is packed into the definition of the limit of a sequence. So the reader is encouraged to take their time and be patient! Here are some pointers to get started:

(a) Most importantly, the condition $|a - a_n| < \varepsilon$ means that a_n *is within a distance of* ε *of* a, i.e., it's in the ball of radius ε centered at a:



If we are dealing with a real sequence, then $|a - a_n| < \varepsilon$ means that a_n is in the interval $(a - \varepsilon, a + \varepsilon) \subset \mathbb{R}$. Note that when a is real, this interval is the intersection of the above ball with the real number line in $\mathbb{C}$.

(b) The rough idea of the limit is that as n get large, a_n gets close to a. By itself, this characterization is too vague. What is meant by "gets close to"? Also, is it OK if some but not all a_n get close to a?

(c) The number ε is a challenge: "Can you make the distance between a_n and a less than ε? The number N is the response: "Yes, if you go out further than N steps in the sequence, then all of the numbers in the sequence past that point are within a distance ε of a."

(d) The number N is a function of ε. If $\varepsilon > 0$ is made smaller, then N usually needs to be made larger—you need to go out further in the sequence to get closer to the limit. So it might be better to write $N(\varepsilon)$ or N_ε instead of just N to highlight this dependence.

(e) This might be very helpful in understanding our definition: $\lim_{n\to\infty} a_n = a$ is equivalent to saying that for every $\varepsilon > 0$, all but a finite number of the a_n are inside the ball of radius ε centered at a.

The following is a template for a limit proof that works directly from the definition of the limit:

Proof Template. $\lim_{n\to\infty} a_n = a$.

Proof. Given $\varepsilon > 0$, let $N =$ blah. Then if $n > N$, we have

$$\begin{aligned} |a - a_n| &= \text{ blah} \\ &= \\ &\leq \\ &= \\ &< \\ &= \varepsilon. \end{aligned}$$

□

For the proof to be valid, at least one strict inequality, "$<$", is required.

Example. Let $a \in \mathbb{C}$, and consider the constant sequence $\{a\}_n = a, a, a, \ldots$. Then $\lim_{n\to\infty} a = a$.

Proof. We are considering the sequence $\{a_n\}$ such that $a_n = a$ for all n. Given $\varepsilon > 0$, let $N = 0$. Then $n > N$ implies

$$|a - a_n| = |a - a| = 0 < \varepsilon.$$

Thus, $\lim_{n\to\infty} a = a$. □

Note that in the above proof, we could have chosen any number $N \in \mathbb{R}$. The distance of *every* term in the sequence is a distance of 0 from a, and $0 < \varepsilon$ by choice of ε.

Example. $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. Given $\varepsilon > 0$, let $N = 1/\varepsilon$. Then $n > N$ implies

$$\begin{aligned} \left|0 - \frac{1}{n}\right| &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} && \text{(since } n > 0\text{)} \\ &< \frac{1}{N} && \text{(since } n > N\text{)} \\ &= \varepsilon && \text{(since } N = 1/\varepsilon\text{).} \end{aligned}$$

□

The above proof is typical in that the value for N is unmotivated. The reader can verify each step of the proof, but may be mystified by the choice of N. That's because one usually constructs a limit proof by starting at the end of the proof (on scratch paper): we want $|a - a_n| < \varepsilon$, and then work backwards to find N. For instance, in this last proof, in the end, we want

$$\left|0 - \frac{1}{n}\right| < \varepsilon.$$

This statement is equivalent to

$$\frac{1}{n} < \varepsilon,$$

which is equivalent to

$$\frac{1}{\varepsilon} < n.$$

Thus, if we take $N = \dfrac{1}{\varepsilon}$ and assume $n > N$, we are OK:

$$\frac{1}{\varepsilon} = N < n.$$

All of our steps are reversible, so things are going to work out.

Example. Let's apply that same reasoning to construct a proof that

$$\lim_{n\to\infty} (1 + 1/\sqrt{n}) = 1.$$

Here $a_n = 1 + 1/\sqrt{n}$ and $a = 1$. In the end, we will want $|a - a_n| < \varepsilon$, i.e.,

$$|1 - (1 + 1/\sqrt{n})| < \varepsilon.$$

On scratch paper we work out

$$\begin{aligned} |1 - (1 + 1/\sqrt{n})| < \varepsilon &\Leftrightarrow |-1/\sqrt{n}| < \varepsilon \\ &\Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon \\ &\Leftrightarrow \frac{1}{\varepsilon} < \sqrt{n} \\ &\Leftrightarrow \frac{1}{\varepsilon^2} < n. \end{aligned}$$

Thus, we can take $N = 1/\varepsilon^2$. Now we are ready to write the formal proof:

Claim. $\lim_{n\to\infty}(1 + 1/\sqrt{n}) = 1$

Proof. Given $\varepsilon > 0$, let $N = 1/\varepsilon^2$ and suppose that $n > N$. It follows that

$$\begin{aligned} \left|1 - \left(1 + \frac{1}{\sqrt{n}}\right)\right| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{N}} && \text{(since } n > N\text{)} \\ &= \varepsilon && \text{(since } N = 1/\varepsilon^2\text{).} \end{aligned}$$

□

(Note that $n > N \Rightarrow \sqrt{n} > \sqrt{N} \Rightarrow \frac{1}{\sqrt{N}} > \frac{1}{\sqrt{n}}$.)