Math 112 lecture for Monday, Week 7

TOPOLOGY

(Supplemental reading: Section 3.5 in Swanson.)

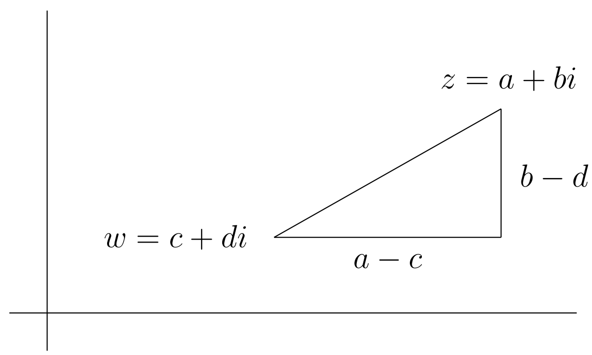
For the following, let $F = \mathbb{R}$ or $\mathbb{C}$.

Definition. The *distance* between $z, w \in F$ is

$$d(z, w) := |z - w|.$$

Example. In $\mathbb{C}$, with $z = a + bi$ and $w = c + di$,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}.$$

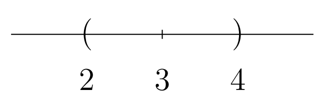


Definition. The *open ball centered at $z \in F$ of radius $r \in \mathbb{R} > 0$* is the subset

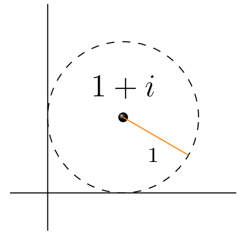
$$B(z; r) := \{w \in F : |w - z| < r\}.$$

Example.

1. If $F = \mathbb{R}$, then what we have just called a ball is an open interval. For example, in $\mathbb{R}$, we have $B(3; 1) = (2, 4)$:

$B(3; 1) = (2, 4)$:  .

2. Here is a picture of the ball of radius 1 in $\mathbb{C}$ centered at $1 + i$:

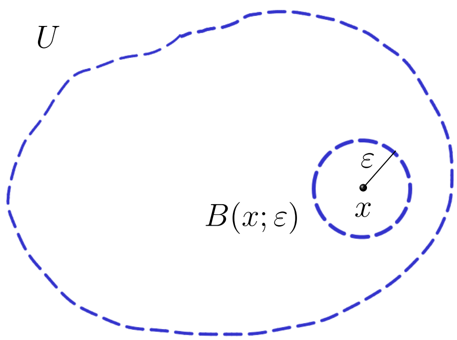
$B(1+i;1):$ 

.

Definition. A subset $U \subseteq F$ is *open* if it contains an open ball about each of its points. This means that for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B(u;\varepsilon) \subseteq U.$$

A key point here is that the open ball must be entirely contained inside U:



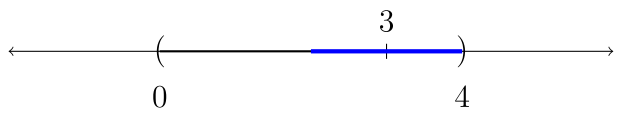
In the above picture, $U \subset \mathbb{C}$ is the set of all points inside the outer dashed line. Inside of U, we show a typical point x and an example of an open ball about x completely contained inside of U. As we test points nearer the border of U, the choice for ε will need to become smaller.

Example.

1. The empty set $\emptyset \subset F$ is open.

2. The set F, itself, is open (whether $F = \mathbb{R}$ or $\mathbb{C}$). Proof: Given $x \in F$, the open ball $B(x;1)$ is contained F.

3. An open interval $(a,b) \subseteq \mathbb{R}$ is open. Given $c \in (a,b)$, let ε be either $c-a$ or $b-c$, whichever is smallest. Then the open interval centered at c and with radius ε is contained in (a,b). For instance, take the interval $(0,4)$, and let $c = 3$. Then $(0,4)$ contains the open ball of radius 1 about c, i.e., it contains the interval $(2,4)$. This open ball is shown in blue below:

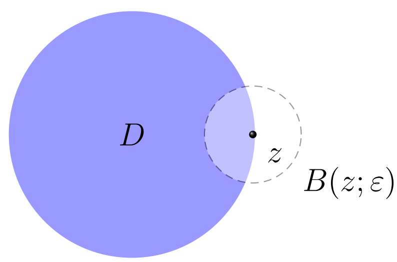


4. A union of open intervals in $\mathbb{R}$ is an open set. In fact, every subset of $\mathbb{R}$ is a union of open intervals. We prove this and the analogous result for $\mathbb{C}$ in a proposition below.

5. In $\mathbb{C} = \mathbb{R}^2$ the Cartesian product $(a, b) \times (c, d)$ is open. (This is a rectangle in $\mathbb{R}^2$ not containing its boundary.)

6. The closed interval $[1, 2)$ is *not* open. The problem is the endpoint 1. We have $1 \in [1, 2]$ but there is no open ball (interval) centered at 1 and contained in $[1, 2)$. Any open interval about 1 will contain points less that 1 and hence not in the set $[1, 2)$.

7. The closed disc $D := \{z \in \mathbb{C} : |z| \leq 1\}$ is not open. The problem is each of the points with modulus 1 sitting on the boundary of D. Any open ball about one of these points will contain points that are not in D, i.e., it will not be completely contained in D:



Proposition. Every open set in F is a union of open balls.

Proof. Let $U \subseteq F$ be open. Since U is open, for each $u \in U$, there exists $\varepsilon_u > 0$ such that

$$B(u; \varepsilon_u) \subseteq U.$$

We claim

$$\bigcup_{u \in U} B(u; \varepsilon_u) = U.$$

To see this, first let $v \in \cup_{u \in U} B(u; \varepsilon_u)$. Then $v \in B(u; \varepsilon_u)$ for some $u \in U$ (in fact, $v \in B(v; \varepsilon_v)$), and since $B(u; \varepsilon_u) \subseteq U$, we have $v \in U$. To see the reverse inclusion let $v \in U$. Then $v \in B(v; ve_v)$, and hence, v is in the union $\cup_{u \in U} B(u; \varepsilon_u)$. $\square$

Proposition. Every nonempty open subset of F has infinitely many elements.

Proof. Let U be a nonempty open subset of F. Since U is nonempty, it contains some point u. Since U is open, it must contain an open ball $B(u;\varepsilon)$ about U. But every open ball contains infinitely many points. For instance $B(u;\varepsilon)$ contains the points $\{u+\varepsilon/n : n = 2, 3, 4, \dots\}$. $\square$

Corollary. No nonempty finite set of points is open. In particular, for each $z \in F$, the set $\{z\}$ is not open.

Definition. The set of open sets of F forms a *topology* on F. This means that

1. $\emptyset$ and F are open.
2. An arbitrary union of open sets is open.
3. A finite intersection of open sets is open.

Regarding part 3, note that an infinite intersection of open sets might not be open. For example,

$$\bigcap_{n=1}^{\infty} B(z; 1/n) = \{z\}$$

which is not open (since it is finite). For a more concrete example, note that

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}\,.$$

Definition. A subset $C \subseteq F$ is *closed* if its complement $F \setminus C$ is open.

Example.

1. F is closed since $F \setminus F = \emptyset$, and $\emptyset$ is open.
2. Similarly $\emptyset$ is closed since $F \setminus \emptyset = F$, and F is open. (Thus, we have seen that F and $\emptyset$ are both open and closed.[1])

[1] If it seems paradoxical to you that there are sets, like the empty set, that are both open and closed, you are not alone (cf. Hitler learns topology where "null set"= "empty set", "neighborhood"="open ball").

3. If $B \subseteq F$ is an open ball, the $F \setminus B$ is closed. For instance, if $F = \mathbb{R}$ and $B = (0; 1)$, then

$$\mathbb{R} \setminus (-1, 1) = (-\infty, -1] \cup [1, \infty)$$

is closed.

4. A closed interval $[a, b]$ in $\mathbb{R}$ is closed. That's because its complement

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$$

is open.

5. If $z \in F$, then the set $\{z\}$ is closed. (Exercise.)

Definition. The *closed ball (disc) of radius r centered at $z \in F$* is

$$D(z; r) := \{w \in F : |w - z| \leq r\}.$$

Example. In $\mathbb{R}$, closed discs are the same as closed intervals $[a, b]$. In $\mathbb{C}$ taking an open ball and adding its boundary gives a closed ball.

The following records an argument explained in the accompanying video lecture:

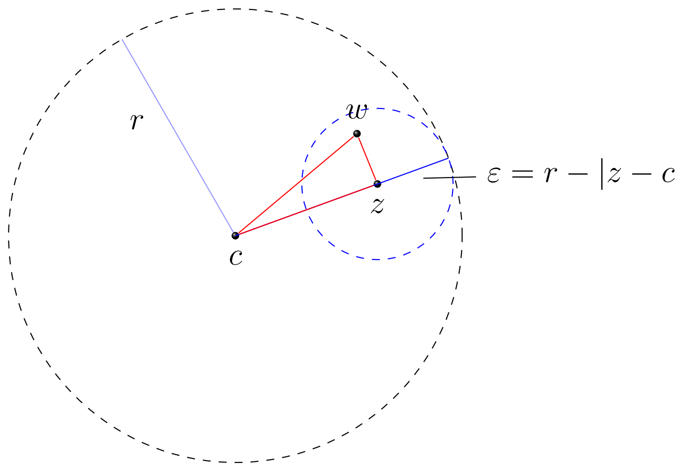
Proposition. An open ball in F is open.

Proof. Let $B(c; r)$ be an open ball in F, and let $z \in B(c; r)$. We must show there is an open ball about z that is completely contained in $B(c; r)$. Let $\varepsilon := r - |z - c|$, and note that $\varepsilon > 0$ since $|z - c| < r$.
We claim that $B(z; \varepsilon) \subseteq B(c; r)$. To see this, $w \in B(z; \varepsilon)$. Then

$$\begin{aligned} d(c, w) = |w - c| &= |(w - z) + (z - c)| \\ &\leq |w - z| + |z - c| \\ &< \varepsilon + |z - c| \\ &= r. \end{aligned}$$

□

Here is a picture motivating the above proof:


r
w
z
c
$\varepsilon = r - |z - c|$