Math 112 lecture for Friday, Week 7

Sequences II

(Supplemental reading: Sections 8.1 and 8.2 in Swanson.)

Recall the definition of the limit of a sequence of complex numbers: we say $\lim_{n\to\infty} a_n = a$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $|a - a_n| < \varepsilon$.

To prove a sequence $\{a_n\}$ doesn't have a limit, we need to show that for all $a \in \mathbb{C}$, the limit of $\{a_n\}$ is not a. To make sense of what this means in terms of the definition of the limit, one pointer is that the negation of "for all" is "there exists", and vice versa: if it is not true that"for $\varepsilon > 0$, there exists $N\ldots$", then there exists $\varepsilon > 0$ such that for all $N\ldots$ The following proposition illustrates this principle.

Proposition. The sequence $\{(-1)^n\} = -1, 1, -1, 1, \ldots$ diverges.

Proof. Let a be any complex number. We claim that $\{(-1)^n\}$ does not converge to a. To see this, let $\varepsilon = 1$, and let N be any real number. There is some even number $n > N$, and for this n,

$$|a - (-1)^n| = |a - 1|.$$

Similarly, these is some odd number $n > N$, and for this n,

$$|a - (-1)^n| = |a + 1|.$$

Next, using the triangle inequality, we see

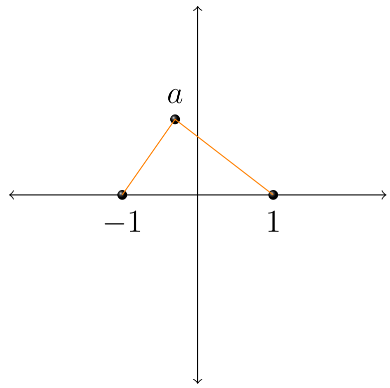
$$2 = |(a-1) - (a+1)| \le |a-1| + |-(a+1)| = |a-1| + |a+1|.$$

Since $2 \le |a-1| + |a+1|$, at least one of $|a-1|$ or $|a+1|$ is greater than or equal to 1. So it follows that there exists some $n > N$ such that

$$|a - (-1)^n| \not< \varepsilon = 1.$$

$\square$

The motivation for the above proof is the following picture:



There is no way to find $a \in \mathbb{C}$ that is both close to -1 and close to 1. In the proof, we applied the triangle inequality to the triangle in the picture to conclude that the distance from a to at least one of 1 or -1 must be at least 1. That leads to taking $\varepsilon = 1$. (The argument works by taking any ε such that $0 < \varepsilon \leq 1$.)

We next prove a result that will be of great use later on. Start with a number $\alpha \in \mathbb{C}$, and consider the sequence $\{\alpha^n\} = \alpha, \alpha^2, \alpha^3, \ldots$ What is the behavior of this sequence? Does it converge? The answer depends on the initial choice of α. The key to unlocking the behavior of the sequence is to consider the polar form of α. Say $\alpha = |\alpha|(\cos(\theta) + i\sin(\theta))$. Then $\alpha^n = |\alpha|^n(\cos(n\theta) + i\sin(n\theta))$. So the length of α^n is $|\alpha|^n$ and the angle (or *argument*) of α^n is n times the angle of α. So it seems clear that if $|\alpha| > 1$, the sequence "diverges to ∞", spinning as it goes (unless the angle of α is 0). Similarly, if $|\alpha| < 1$, the sequence is sucked into the origin. What if $|\alpha| = 1$? Then α is a point on the unit circle, and unless $\arg(\alpha) = 0$, the sequence will perpetually spin around the circle, never converging. See Figure 1.

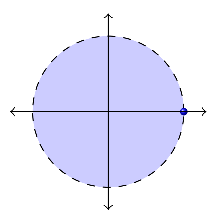


Figure 1: The sequence $\{\alpha^n\}$ converges to 1 if $\alpha = 1$, to 0 if $|\alpha| < 0$, and diverges, otherwise. The α for which the sequence converges appear in blue.

Proposition. Let $\alpha \in \mathbb{C}$ and consider the sequence $\{\alpha_n\}$. There are three cases:

(a) $\lim_{n\to\infty} \alpha^n = 1$ if $\alpha = 1$;

(b) $\lim_{n\to\infty} \alpha^n = 0$ if $|\alpha| < 1$;

(c) $\{\alpha^n\}$ diverges, otherwise, i.e., if $\alpha \neq 1$ and $|\alpha| \geq 1$.

Proof. For case (a), where $\alpha = 1$, we get the constant sequence $1, 1, 1, \ldots$ Last time we saw that the limit of a constant sequence is just the constant (given $\varepsilon > 0$, we can take N to be any real number).

For case (b), suppose that $|\alpha| < 1$. If $\alpha = 0$, then α^n is the constant sequence $0, 0, 0, \ldots$, with limit 0. Otherwise, given $\varepsilon > 0$, we need to find $N \in \mathbb{R}$ such that if $n > N$, then

$$|0 - \alpha^n| < \varepsilon.$$

This is equivalent to showing that

$$|\alpha|^n < \varepsilon.$$

To find N, we solve for n in the above equation:

$$|\alpha|^n < \varepsilon \quad \Leftrightarrow \quad \ln(|\alpha|^n) < \ln(\varepsilon) \quad \Leftrightarrow \quad n \ln(|\alpha|) < \ln(\varepsilon) \quad \Leftrightarrow \quad n > \ln(\varepsilon)/\ln(|\alpha|).$$

There are a couple of subtleties in the above calculation. In the first step, we used the fact that taking logs preserves inequalities. This is because the log is an increasing function: $\ln(x) < \ln(y)$ if and only if $x < y$. In the last step of the calculation, the inequality is reversed since $|\alpha| < 1$ means that $\ln(|\alpha|) < 0$. To complete the proof that $\lim_{n\to\infty} \alpha^n = 0$ in this case, let $N := \ln(\varepsilon)/\ln(|\alpha|)$ and suppose that $n > N$. Running the string of implications displayed above in reverse, we see that it follows that $n > N$ implies

$$|0 - \alpha^n| = |\alpha|^n < \varepsilon,$$

as required.

The last case is the toughest. Suppose that $\alpha \neq 1$ and $|\alpha| \geq 1$. We must show that $\{\alpha^n\}$ is divergent. We prove this by contradiction. Suppose that $\lim_{n\to\infty} \alpha^n = \lambda$ for some $\lambda \in \mathbb{C}$. Define

$$\varepsilon := \frac{|\alpha - 1|}{2}.$$

Note that $\varepsilon > 0$ since $\alpha \neq 1$. We'll see below that this particular ε will give us insurmountable problems, forcing the desired contradiction. Since $\lim_{n\to\infty} \alpha^n = \lambda$, there exists $N \in \mathbb{R}$ such that

$$n > N \quad \Longrightarrow \quad |\lambda - \alpha^n| < \varepsilon = \frac{|\alpha - 1|}{2}.$$

Now, if $n > N$, it follows that $n + 1 > N$, too. Hence,

$$|\lambda - \alpha^{n+1}| < \frac{|\alpha - 1|}{2}.$$

Therefore, if $n > N$, it follows that

$$\begin{aligned}
|\alpha^{n+1} - \alpha^n| &= |(\lambda - \alpha^n) - (\lambda - \alpha^{n+1})| \\
&\leq |\lambda - \alpha^n| + |\lambda - \alpha^{n+1}| && \text{(triangle inequality)} \\
&< \frac{|\alpha - 1|}{2} + \frac{|\alpha - 1|}{2} \\
&= |\alpha - 1|,
\end{aligned}$$

i.e.,

$$|\alpha^{n+1} - \alpha^n| < |\alpha - 1|$$

for all $n > N$. On the other hand,

$$\begin{aligned}
|\alpha^{n+1} - \alpha^n| &= |\alpha|^n |\alpha - 1| \\
&\geq |\alpha - 1| && \text{since } |\alpha| \geq 1.
\end{aligned}$$

Thus, our assumption that $\lim_{n\to\infty} \alpha^n = \lambda$ has allowed us to show that there is an N such that $n > N$ implies both

$$|\alpha^{n+1} - \alpha^n| < |\alpha - 1|$$

and

$$|\alpha^{n+1} - \alpha^n| \geq |\alpha - 1|.$$

That's not impossible. So there is no such λ. $\square$

Example. The sequence $\left\{\left(\frac{1+i}{2}\right)^n\right\}$ converges since

$$\left|\frac{1+i}{2}\right| = \frac{1}{2}|i+1| = \frac{1}{2}\sqrt{1^2+1^2} = \frac{\sqrt{2}}{2} < 1.$$

The sequence $\{i^n\}$ does not converge since $|i| = 1$ and $i \neq 1$. A picture of the first sequence appears below: