Math 112 lecture for Wednesday, Week 6

Complex numbers II

(Supplemental reading: Sections 3.2 and 3.3 in Swanson.)

$\mathbb{C}$ cannot be ordered. It turns out that the field $\mathbb{C}$ cannot be ordered. To see this, recall we showed that if x is any nonzero element in an ordered field, then $x^2 > 0$. In particular, letting $x = 1$, we saw that $1 > 0$, and then that $-1 < 0$. Suppose $\mathbb{C}$ could be ordered. Then, since i is nonzero, we would have that $i^2 > 0$, but in fact, $i^2 = -1 < 0$. By the trichotomy axiom, we cannot have both $i^2 > 0$ and $i^2 < 0$.

The triangle inequality for complex numbers. Our goal today is to prove the triangle inequality for the complex numbers. You may recall that we already proved the triangle inequality for any ordered field F:

$$|x + y| \leq |x| + |y|$$

for all $x, y \in F$. A problem arises immediately in trying to extend this result to the complex numbers: our definition of $|x|$ depended on F being ordered, yet, $\mathbb{C}$ can not be ordered. So we start with a new definition of "absolute value" that will work in the context of complex numbers.

Definition. Let $z = a + bi \in \mathbb{C}$. The *conjugate* of z is

$$\overline{z} := a - bi.$$

The *modulus* or *length* of z is

$$|z| := \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}.$$

The *real part* of z is

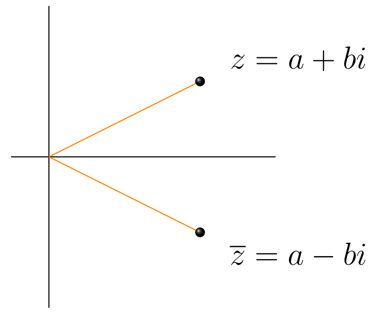
$$\mathrm{Re}(z) := a$$

and the *imaginary part* of z is

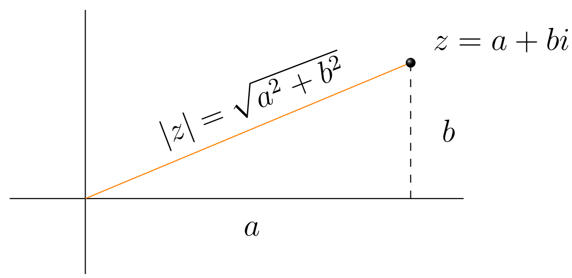
$$\mathrm{Im}(z) := b.$$

Remarks.

1. Geometrically, taking the conjugate amounts to flipping about the x-axis (also known as the *real axis* in the context of complex numbers):



2. The modulus of a complex number is the distance of the number from the origin:



3. Note that both the real and imaginary part of a complex number are real numbers. For instance $\mathrm{Im}(2 + 3i) = 3 \in \mathbb{R}$.

4. If $r \in \mathbb{R}$, we already have a definition of its absolute value

$$|r| := \begin{cases} r & \text{if } r \geq 0, \\ -r & \text{if } r < 0. \end{cases}$$

However, we may now consider $r = r + 0 \cdot i$ as a complex number and compute its modulus:

$$|r| = \sqrt{r^2 + 0^2} = \sqrt{r^2}.$$

Note that the expression on the right, $\sqrt{r^2}$, is the ordinary absolute value of r as a real number. So there is no conflict between $\mathbb{R}$ and $\mathbb{C}$ in our notation for $|r|$.

Practice problems. (Solutions appear on the last page.)

1. Let $z = 4 + 2i + (5 + i)i$. Find $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

2. Let $z = 4 + 5i$. What is $\overline{z}$? What is $|z|$? What are $z + \overline{z}$ and $z - \overline{z}$?

Proposition. Let $z, w \in \mathbb{C}$. Then

1. $\overline{\overline{z}} = z$.
2. $|\overline{z}| = |z|$.
3. $\overline{z+w} = \overline{z} + \overline{w}$.
4. $\overline{zw} = \overline{z}\,\overline{w}$.
5. $|zw| = |z||w|$.
6. $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
7. $z + \overline{z} = 2\mathrm{Re}(z)$ and $z - \overline{z} = 2i\,\mathrm{Im}(z)$.
8. (triangle inequality for complex numbers) $|z+w| \leq |z| + |w|$.
9. If $z \neq 0$, then $z/|z|$ is a *unit vector*, i.e.,

$$\left|\frac{z}{|z|}\right| = 1.$$

Proof.

1. Let $z = a + bi$. Then

$$\overline{\overline{z}} = \overline{(\overline{a+bi})} = \overline{a - bi} = a + bi = z.$$

2. Using part 1,

$$|\overline{z}| = \sqrt{\overline{z}\,\overline{\overline{z}}} = \sqrt{\overline{z}\,z} = \sqrt{z\,\overline{z}} = z.$$

3. Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}
\overline{z+w} &= \overline{(a+bi)+(c+di)} \\
&= \overline{(a+c)+(b+d)i} \\
&= (a+c)-(b+d)i \\
&= (a-bi)+(c-di) \\
&= \overline{z} + \overline{w}.
\end{aligned}$$

4. Exercise.
5.

$$|zw| = \sqrt{(zw)\overline{zw}} = \sqrt{(zw)(\overline{z}\overline{w})} = \sqrt{(z\overline{z})(w\overline{w})} = \sqrt{z\overline{z}}\sqrt{w\overline{w}} = |z||w|.$$

6. If $z = a + bi$, we have

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{a^2} = |a| = |\mathrm{Re}(z)|.$$

The result for the complex part of z is similar.
7. Let $z = a + bi$. Then

$$z + \overline{z} = (a + bi) + (a - bi) = 2a = 2\mathrm{Re}(z),$$

and

$$z - \overline{z} = (a + bi) - (a - bi) = 2bi = 2i\,\mathrm{Im}(z).$$

8.

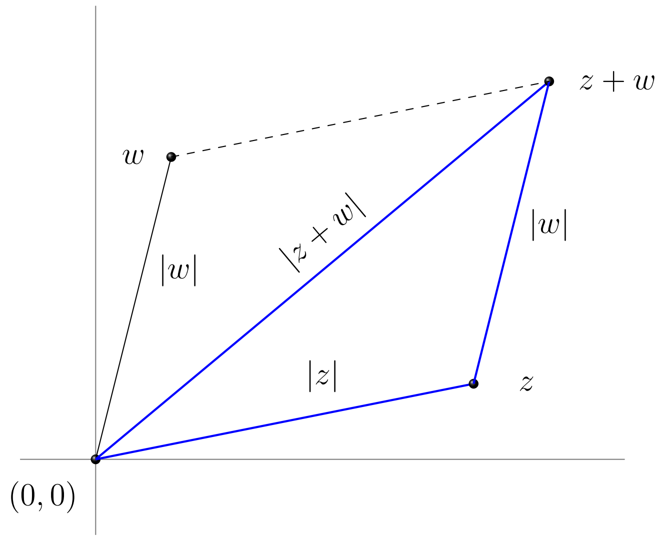
$$\begin{aligned}
|z + w|^2 &= (z + w)(\overline{z + w}) && \text{(definition of modulus)} \\
&= (z + w)(\overline{z} + \overline{w}) && \text{(part 3)} \\
&= z\,\overline{z} + z\,\overline{w} + \overline{z}\,w + \overline{w}\,w \\
&= |z|^2 + z\,\overline{w} + \overline{z}\,w + |w|^2 \\
&= |z|^2 + z\,\overline{w} + \overline{z}\,\overline{\overline{w}} + |w|^2 && \text{(part 1)} \\
&= |z|^2 + z\,\overline{w} + \overline{z\overline{w}} + |w|^2 && \text{(part 4)} \\
&= |z|^2 + 2\mathrm{Re}(z\,\overline{w}) + |w|^2 && \text{(part 6)} \\
&\leq |z|^2 + 2|\mathrm{Re}(z\,\overline{w})| + |w|^2 \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 && \text{(part 5)} \\
&= |z|^2 + 2|z||\overline{w}| + |w|^2 && \text{(part 4)} \\
&= |z|^2 + 2|z||w| + |w|^2 && \text{(part 2)} \\
&= (|z| + |w|)^2\,.
\end{aligned}$$

Now take square roots to get the result.
9.

$$\left|\frac{z}{|z|}\right| = \left|z \cdot \frac{1}{|z|}\right| = |z|\left|\frac{1}{|z|}\right| = |z|\frac{1}{|z|} = 1.$$

$\square$

The picture below may help to give an intuitive understanding of the triangle inequality.



Solutions to practice problems.

1. We have
$$z = 4 + 2i + (5+i)i = 4 + 2i + 5i - 1 = 3 + 7i.$$
Hence, $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = 7$.

2. If $z = 4 + 5i$, then
$$\begin{aligned} \overline{z} &= 4 - 5i \\ |z| &= \sqrt{4^2 + 5^2} = \sqrt{41} \\ z + \overline{z} &= 4 + 5i + 4 - 5i = 8 = 2\,\mathrm{Re}(z) \\ z - \overline{z} &= 4 + 5i - (4 - 5i) = 10i = 2i\,\mathrm{Im}(z). \end{aligned}$$