Math 112 lecture for Monday, Week 6

Complex numbers I

(Supplemental reading: Section 3.1 in Swanson.)

Definition. The *complex numbers* or *complex plane*, denoted $\mathbb{C}$, is the set $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R} = \{(a,b) : a,b \in \mathbb{R}\}$ with addition defined by

$$(a,b) + (c,d) := (a+c, b+d),$$

and multiplication defined by

$$(a,b)(c,d) := (ac - bd, ad + bc)$$

for all $(a,b), (c,d) \in \mathbb{C}$.

Thus, the complex numbers are the ordinary plane, $\mathbb{R}^2$, with vector addition and a "twisted" multiplication. We have added algebraic operations to the geometric object $\mathbb{R}^2$ to create $\mathbb{C}$.

Example. We have

$$(2,3) + (-4,6) = (-2,9)$$

and

$$(2,3)(-4,6) = (2(-4) - 3 \cdot 6,\ 2 \cdot 6 + 3(-4)) = (-26, 0).$$

Proposition. $\mathbb{C}$ is a field.

Proof. The verification is left to the reader with some hints provided below:

Addition in $\mathbb{C}$ is commutative. To see this, let $(a,b), (c,d) \in \mathbb{C}$. Then

$$\begin{aligned}
(a,b) + (c,d) &= (a+c, b+d) && \text{(definition of addition in } \mathbb{C}\text{)} \\
&= (c+a, d+b) && \text{(addition in } \mathbb{R} \text{ is commutative)} \\
&= (c,d) + (d,c) && \text{(definition of addition in } \mathbb{C}\text{).}
\end{aligned}$$

The additive identity for $\mathbb{C}$ is $(0,0)$: for all $(a,b) \in \mathbb{C}$,

$$(a,b) + (0,0) := (a+0, b+0) = (a,b).$$

The multiplicative identity for $\mathbb{C}$ is $(1,0)$: for all $(a,b) \in \mathbb{C}$,

$$(a,b)(1,0) := (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a,b).$$

Nonzero elements of $\mathbb{C}$ have multiplicative inverses: for all $(a,b) \in \mathbb{C}$, we have

$$\begin{aligned}
(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) &= \left(a \cdot \frac{a}{a^2+b^2} - b \cdot \frac{-b}{a^2+b^2}, a \cdot \frac{-b}{a^2+b^2} + b \cdot \frac{a}{a^2+b^2}\right) \\
&= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}\right) \\
&= (1,0)\,.
\end{aligned}$$

□

Algebraic notation for complex numbers. Here we want to introduce the ordinary algebraic notation for complex numbers as an alternative to the "geometric" notation we used in our definition. There is an injection

$$\begin{aligned}
s\colon \mathbb{R} &\to \mathbb{C} \\
x &\mapsto (x,0)
\end{aligned}$$

and this injection "respects addition" meaning that for all $x, y \in \mathbb{R}$, we have $s(x+y) = s(x) + s(y)$:

$$s(x+y) = (x+y,0) = (x,0) + (y,0) = s(x) + s(y).$$

Similarly, s respects multiplication: for all $x, y \in \mathbb{R}$,

$$s(xy) = (xy,0) = (x,0)(y,0) = s(x)s(y).$$

(The reader should check that $(xy,0) = (x,0)(y,0)$ using the definition of multiplication for $\mathbb{C}$.)

The injection s allows us to think of the real numbers as the x-axis in the complex plane. In this way it is natural to just write x for the complex number $(x,0)$. Thus, for example, in the context of the complex numbers, we will think of the real number $\sqrt{2}$ as the point $(\sqrt{2},0)$ in the plane. Since s preserves addition and multiplication, no conflict will arise in doing arithmetic with elements in $\mathbb{R}$ versus with their corresponding elements in the complex plane.

You may have seen complex numbers written in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. In order to understand that notation, the reader should first verify the following: for each $(a,b) \in \mathbb{C}$,

$$(a,b) = (a,0) + (b,0)(0,1). \tag{1}$$

As explained above, for every real number x, we write x in place of $(x,0) \in \mathbb{C}$, and now we define

$$i := (0,1).$$

With this convention, (1) becomes

$$(a, b) = (a, 0) + (b, 0)(0, 1) = a + bi.$$

Note that $(1, 0)$ is then just written as 1 and $(0, 0)$ is written as 0, which is perfect since $(1, 0)$ is the multiplicative identity and $(0, 0)$ is the additive identity of $\mathbb{C}$.

To see that it is reasonable to think of $i = (0, 1)$ as a square root of -1, we square i:

$$i^2 := (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1.$$

Let's be careful with that last step, $(-1, 0) = -1$. The notation -1 means "the thing we add to the multiplicative identity to get the additive identity". In our case, the multiplicative identity is $(1, 0)$ and the additive identity is $(0, 0)$, and we have

$$(-1, 0) + (1, 0) = (-1 + 1, 0 + 0) = (0, 0) = 0.$$

So $i^2 = -1$, i.e., i^2 is the additive inverse of the multiplicative identity of $\mathbb{C}$.

The notation $a+bi$ is better suited for arithmetic calculations. The rule $(a, b)+(c, d) = (a + b, c + d)$ becomes

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and the rule $(a, b)(c, d) = (ac - bd, ad + bc)$ becomes

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd(-1) \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

In other words, we manipulate these symbols as ordinary algebraic expressions but with the rule $i^2 = -1$.

So we have two ways of thinking about complex numbers: geometrically, as points (a, b) in the plane, and algebraically as expressions $a + bi$.

Examples. Here are some examples featuring the algebraic notation.

1. $(5 - 6i) + 4(2 + 5i) = (5 - 6i) + (8 + 20i) = 13 + 14i$.

2. $(4 + 3i)(1 + i) + (3 + 2i) = (1 + 7i) + (3 + 2i) = 4 + 9i$.

3. Since $\mathbb{C}$ is a field, every nonzero complex number has a multiplicative inverse. As with any field, if z is a nonzero element, its inverse is denote $1/z$. In this example, we will compute

$$\frac{1}{3+5i},$$

the multiplicative inverse of $3+5i$. There is an **important conjugation trick** for computing this: multiply the top and bottom by the *conjugate*, $3-5i$:

$$\begin{aligned}\frac{1}{3+5i} &= \frac{1}{3+5i}\cdot\frac{3-5i}{3-5i}\\ &= \frac{3-5i}{(3+5i)(3-5i)}\\ &= \frac{3-5i}{3^2+5^2}\\ &= \frac{3}{34}-\frac{5}{34}i.\end{aligned}$$

In the end, we are able to express the multiplicative inverse of $3+5i$ in the form $a+bi$ with $a,b\in\mathbb{R}$. Recall that earlier, using the geometric notation, we found that the inverse of a general nonzero $(a,b)\in\mathbb{C}$ is

$$\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right).$$

In algebraic notation, this is

$$\frac{1}{a+bi} = \frac{1}{a+bi}\cdot\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i,$$

which agrees with what we just found in the case of $3+5i$.

4. Here, we use the conjugation trick to write $\frac{2+4i}{6+i}$ in the form $a+bi$ with $a, b \in \mathbb{R}$:

$$\begin{aligned}
\frac{2+4i}{6+7i} &= \frac{2+4i}{6+7i} \cdot \frac{6-7i}{6-7i} \\
&= \frac{(2+4i)(6-7i)}{(6+7i)(6-7i)} \\
&= \frac{40+10i}{6^2+7^2} \\
&= \frac{40+10i}{85} \\
&= \frac{8+2i}{17} \\
&= \frac{8}{17} + \frac{2}{17}i.
\end{aligned}$$