Math 112 lecture for Friday, Week 6

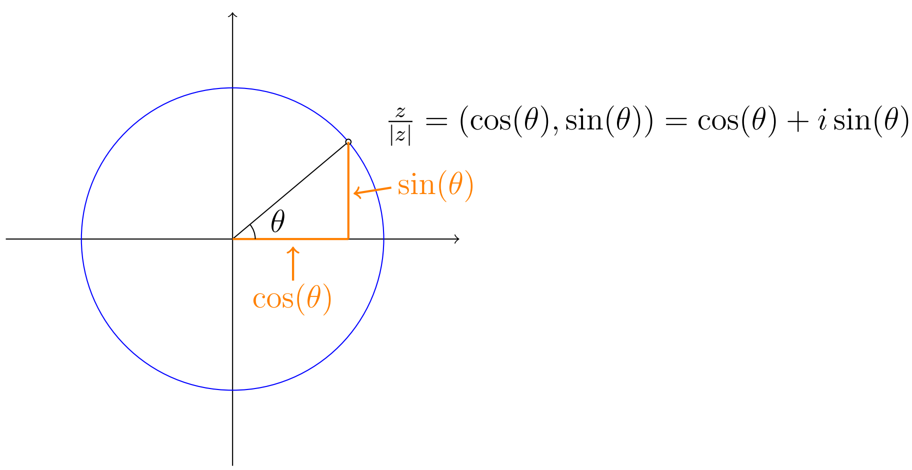
Complex numbers III

(Supplemental reading: Section 3.4 in Swanson.)

polar form for a complex numbers

Let z be a nonzero complex number. Then its length, $|z|$, is a nonzero real number, and we can write
$$z = |z| \cdot \frac{z}{|z|}.$$
The number $z/|z|$ has unit length:
$$\left|\frac{z}{|z|}\right| = \left|z \cdot \frac{1}{|z|}\right| = |z| \cdot \left|\frac{1}{|z|}\right| = |z| \cdot \frac{1}{|z|} = 1.$$

So $z/|z|$ is a point in the plane with unit length, i.e., it's a point on the unit circle:



Therefore, we can write
$$\frac{z}{|z|} = (\cos(\theta), \sin(\theta)) = \cos(\theta) + i\sin(\theta)$$
for some angle θ.

Definition. The *angle* or *argument* of $z \in \mathbb{C} \setminus \{0\}$ is
$$\arg(z) := \theta$$

where θ is the angle shown above. The *polar form* for z is

$$z = |z|(\cos(\theta) + i\sin(\theta)).$$

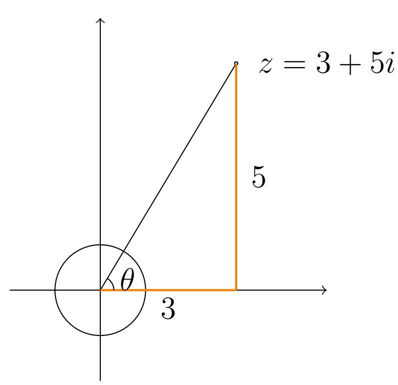
The polar form describes z in terms of its length $|z|$ and angle θ.

Examples.

1. Let $z = 7 + 7i$. Then
$$|z| = \sqrt{7^2 + 7^2} = 7\sqrt{2}$$
   and
$$\frac{z}{|z|} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$
   Plotting this complex number, it's clear angle is $45^\circ = \pi/4$. So the polar form for z is
$$z = 7\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)i\right).$$

2. Let $z = \sqrt{3} + i$. The length of z is
$$|z| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{3+1} = 2.$$
   Hence,
$$\frac{z}{|z|} = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$
   You should notice that $(\sqrt{3}/2, 1/2)$ is the point on the unit circle with angle $30^\circ = \pi/6$. Hence, $\arg(z) = \pi/6$, and the polar form for z is
$$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)i\right).$$

3. Let $z = 3 + 5i$:



The length of z is

$$|z| = \sqrt{3^2 + 5^2} = \sqrt{34},$$

the angle of z is

$$\arg(z) = \tan^{-1}\left(\frac{5}{3}\right) \approx 59^\circ \approx 1.03\,\text{rad}.$$

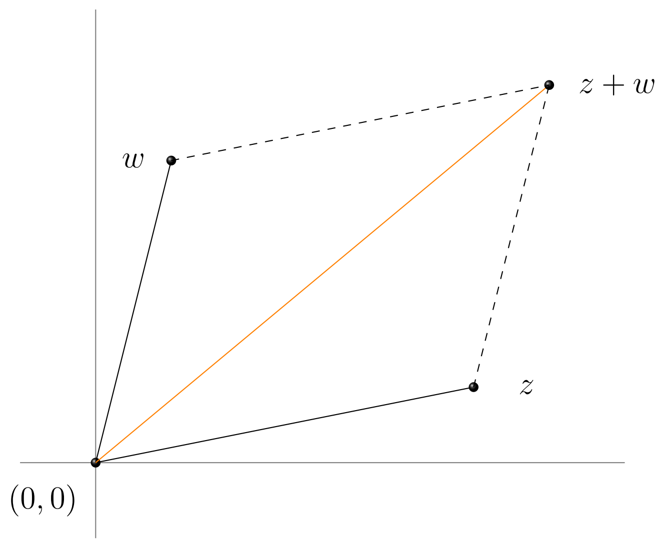
Therefore, the polar form for z is

$$z = \sqrt{34}(\cos(\theta) + \sin(\theta)i)$$

where $\theta = \tan^{-1}(5/3)$.

GEOMETRY OF ADDITION AND MULTIPLICATION IN $\mathbb{C}$

The geometry of addition in $\mathbb{C}$ is just the geometry of vector addition. It is given by the "parallelogram rule":



To see the geometry of multiplication, the secret is to use polar form. Given two nonzero complex numbers $z, w \in \mathbb{C}$, write each in polar form

$$z = |z|\left(\cos(\theta) + \sin(\theta)i\right)$$
$$w = |w|\left(\cos(\psi) + \sin(\psi)i\right).$$

Their product is then

$$zw = |z|\left(\cos(\theta) + \sin(\theta)i\right)|w|\left(\cos(\psi) + \sin(\psi)i\right)$$

$$
\begin{aligned}
&= |z||w| \left(\cos(\theta) + \sin(\theta)i\right) \left(\cos(\psi) + \sin(\psi)i\right) \\
&= |zw| \left(\cos(\theta) + \sin(\theta)i\right) \left(\cos(\psi) + \sin(\psi)i\right) \\
&= |zw| \left(\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi) + \left(\cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta)\right) i\right) \\
&= |zw| \left(\cos(\theta + \psi) + \sin(\theta + \psi)i\right).
\end{aligned}
$$

The last step uses the angle sum formulas for cosine and sine. The last line gives the polar form for zw and reveals the geometry: when multiplying two complex numbers, **angles add and lengths multiply**.

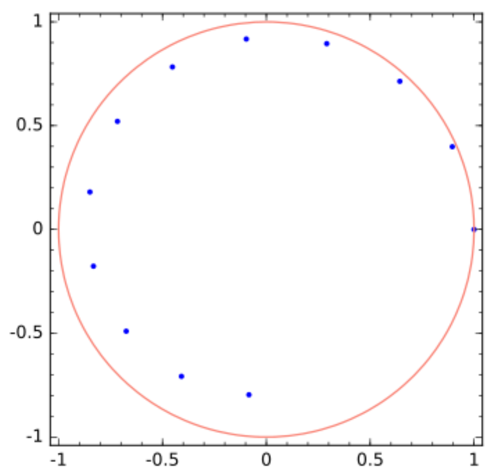
Example. Consider the complex number

$$\alpha = 0.95 \left(\cos\left(\frac{2\pi}{15}\right) + \sin\left(\frac{2\pi}{15}i\right)\right).$$

What will the sequence of complex numbers

$$1 = \alpha^0,\ \alpha,\ \alpha^2,\ \alpha^3,\ \ldots$$

look like in the plane? The length of α is 0.95, and so the length of $\alpha^n = (0.95)^n$. Since 0.95 is less than 1, its powers will get smaller and smaller, which means that this sequence of points will get closer and closer to the origin. The argument of α is an angle that is 1/15-th of the circle. To get to each successive number in the sequence, we multiply by α, which adds that angle to each succesive number, and scale by 0.95. So the sequence spirals around the origin, getting closed to the origin with each step.



The Sage code on cocalc.com that produced this image is on the next page.

Code on cocalc.com:

```
s = 15
a = 0.98*exp(I*2*pi/s)
p = list_plot([a^n for n in range(12)])
q = parametric_plot\left( cos(x),sin(x)),(x,0,2*pi),color='salmon')
(p+q).show(aspect_ratio=1,axes=false,frame=true)
```