Math 112 lecture for Wednesday, Week 5

Completeness

(Supplemental reading: Example 2.7 in Swanson.)

We now come to the last axiom we need to characterize the real numbers: completeness. Let F be any ordered field, and let $S \subseteq F$. To state the axiom, we first need some vocabulary:

- $B \in F$ is an *upper bound* for S if $s \leq B$ for all $s \in S$,
- $b \in F$ is an *lower bound* for S if $b \leq s$ for all $s \in S$,
- S is *bounded* if it has both an upper bound and a lower bound.

Example. Let $S := (0, 1) := \{x \in \mathbb{R} : 0 < x < 1\}$, a subset of the ordered field $\mathbb{R}$. Then any number greater than or equal to 1 is an upper bound for S. For example, 1, π, 42, and 10^6 are all upper bounds. Similarly, any number less than or equal to 0 is a lower bound. For example, $0, -1, -\pi$ are all lower bounds. Not every set has a lower bound. For example, if $S = \mathbb{Z}$, the integers, then S is a subset of the ordered field $\mathbb{R}$ which has neither an upper bound nor a lower bound.

Some more vocabulary:

- $B \in F$ is a *supremum* for S if it is a *least upper bound*. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$. If B exists, then we write $B = \sup(S)$ or $B = \mathrm{lub}(S)$.
- $b \in F$ is an *infimum* for S if it is a *greatest lower bound*. This means that b is a lower bound and if b' is any lower bound, then $b' \leq b$. If b exists, then we write $b = \inf(S)$ or $b = \mathrm{glb}(S)$.

Examples.

1. If $S = (0, 1) \subset \mathbb{R}$, then $\sup(S) = 1$ and $\inf(S) = 0$.

   Important: Note that, as in this example, *the supremum and infimum of a set are not necessarily elements in the set.*

2. Let $S = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\} \subset \mathbb{R}$. Then $\sup(S) = 1$ and $\inf(S) = 0$, as before. However, this time $\inf(S) \in S$ while $\sup(S) \notin S$.

3. If $S = (-2, \infty) = \{x \in \mathbb{R} : -2 < x\} \subset \mathbb{R}$, then $\sup S$ does not exist and $\inf S = -2$.

4. If $S = \{1/n : n = 1, 2, 3, \dots\} \subset \mathbb{R}$, then $\sup(S) = 1 \in S$ and $\inf(S) = 0 \notin S$. Note that in this example, finding the infimum is a way to take a limit of the sequence of rationals $1, 1/2, 1/3, 1/4, \dots$

We've just seen that if a set S has a sup or inf, then it may or may not be the case that either is contained in S. This leads to a bit more vocabulary: if S has a supremum B and $B \in S$, then we call B the *maximum* or *maximal element* of S and write $\max(S) = B$. Similarly, if S has an infimum b and $b \in S$, then we call b the *minimum* of *minimal element* of S and write $\min(S) = b$.

Example. If $S = (0, 1] \subset \mathbb{R}$, then $\max(S) = 1$ and $\min(S)$ does not exist and $\inf(S) = 0$.

Definition. An ordered field F is *complete* if every *nonempty* subset of F which is bounded above has a supremum.

Consider the ordered field of rational numbers, $\mathbb{Q}$, and pick any sequence of rational numbers converging to π, say by just truncating the decimal expansion, and put these numbers into a set

$$S = \{3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ \dots\}.$$

Each of the numbers in S is rational, e.g., $3.14 = \frac{314}{100}$. Does this set of rational numbers have a supremum? The answer depends on which field we are working in. In $\mathbb{R}$, the answer is "yes": $\sup S = \pi$ (and it's not in S). However, in $\mathbb{Q}$ it does not have a supremum. To see this, suppose that $B \in \mathbb{Q}$. If $\pi < B$, then there is a rational number B' such that $\pi < B' < B$.[1] So B' is an upper bound for S that is smaller than B. Therefore, in this case B is not a least upper bound. On the other hand, if $B < \pi$, then there is an element $s \in S$ such that $B < s$ (since we can get arbitrarily close to π by taking the decimal expansion for π and truncating it sufficiently far out). In this case, B is not an upper bound for S. The rational numbers have this "defect": there are nonempty subsets of $\mathbb{Q}$ that are bounded above but have no least upper bound. Thus, both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields, but of these two, only $\mathbb{R}$ also satisfies the completeness axiom. (Of course, we have not *proved* these properties of $\mathbb{Q}$ and $\mathbb{R}$ since, in fact, we have not even given definitions for either of these field. Instead, we are appealing to the reader's prior experience with these number systems.)

[1] One way to create B' is to go out in the decimal expansion for π far enough to get a number much closer to π than to B, then slightly round up that number by adding 1 to the last decimal place. The resulting number will be slightly bigger than π and less than B.

Challenge. Show that an order field is complete if and only if every nonempty subset of F which is bounded below has an infimum.

Theorem. There exists a unique complete ordered field. By *uniqueness*, we mean that if F_1 and F_2 are complete ordered fields, then there exists a bijection $g\colon F_1 \to F_2$ such that for all $a, b \in F_1$,

1. $g(a+b) = g(a) + g(b)$,
2. $g(ab) = g(a)g(b)$,
3. if $a > 0$, then $g(a) > 0$.

Proof. We will accept this result on faith for now and possibly return to it later. $\square$

Since g is a bijection, we can think of it as simply a relabeling of the elements of F_1: each element $a \in F_1$ is now called $g(a) \in F_2$, instead. The listed requirements for g amount to saying that F_1 and F_2 are the same ordered fields up to relabeling.

We finally get to our ultimate goal:

Definition. The complete ordered field is called the field of *real numbers*, denoted $\mathbb{R}$.

While we have not given a construction of the real numbers, the previous theorem says that anything we can prove about the real numbers may be derived from the fourteen field axioms, the four order axioms, and the completeness axiom (i.e., that every nonempty subset that is bounded above has a supremum).

Important vocabulary. We will use the vocabulary introduced in this lecture extensively in the second half of this course. Here is a summary: First there are properties pertaining to subsets of an ordered field: upper bound, lower bound, bounded, supremum, infimum, maximum, and minimum. Second, there is a property an ordered field may possess: completeness. Please take a few moments now to review these terms, trying to come up with your own examples.