Math 112 lecture for Monday, Week 5

## Order axioms

(Supplemental reading: Example 2.7 in Swanson.)

The real numbers have more structure than that described by the field axioms. For instance, we know that, for instance, $\sqrt{2} < 6$, which does not follow from just the field axioms. What does “$<$” even mean here? A complete answer to that question would require a rigorous definition of $\mathbb{R}$. Instead, we first summarize the essential properties.

**Definition.** An *ordered field* is a field $F$ with a relation, denoted $<$, satisfying

**O1.** (Trichotomy) For all $x, y \in F$, exactly one of the following statements is true:

$$x < y, \quad y < x, \quad x = y.$$

**O2.** (Transitivity) The relation $<$ is transitive, i.e., for all $x, y, z \in F$,

$$x < y \quad \text{and} \quad y < z \quad \Longrightarrow \quad x < z.$$

**O3.** (Additive translation) For all $x, y, z \in F$,

$$x < y \quad \Longrightarrow \quad x + z < y + z.$$

**O4.** (Multiplicative translation) For all $x, y, z \in F$,

$$x < y \quad \text{and} \quad 0 < z \quad \Longrightarrow \quad xz < yz.$$

**Remark.** We write $x > y$ if $y < x$, and we write $x \leq y$ if either $x = y$ or $x < y$, and so on.

In the next proposition, we list several properties of $<$ with which you are familiar in the context of $\mathbb{Q}$ or $\mathbb{R}$. We will see that these properties follow from the order axioms, alone.

**Proposition 1.** Let $x, y, z, w$ be elements of an ordered field $F$. Then

1. $x < y \Rightarrow -y < -x$.

2. $x < y$ and $z < 0 \Rightarrow xz > yz$.

3. $x^2 > 0$ if $x \neq 0$.

4. $1 > 0$ and $-1 < 0$.

5. $w < x$ and $y < z \Rightarrow w + y < x + z$.

*Proof.* 1. We have

$$\begin{aligned}
x < y \quad &\Rightarrow \quad -x + x < -x + y && \text{(additive translation)}\\
&\Rightarrow \quad -y + (-x + x) < -y + (-x + y) && \text{(additive translation)}\\
&\Rightarrow \quad -y + (-x + x) < (-y + y) + (-x) && \text{(assoc. and commutativity of +)}\\
&\Rightarrow \quad -y + 0 < 0 + (-x) && \text{(def. of } -x \text{ and } -y\text{)}\\
&\Rightarrow \quad -y < -x && \text{(0 is the additive identity).}
\end{aligned}$$

2. If $z < 0$, then $-z > 0$ by part 1. Then, by multiplicative translation,

$$\begin{aligned}
x < y \quad &\Rightarrow \quad (-z)x < (-z)y && \\
&\Rightarrow \quad -(zx) < -(zy) && \text{(exercise: } (-a)b = -(ab)\text{)}\\
&\Rightarrow \quad -(-(zx) > -(-(zy)) && \text{(by part 1)}\\
&\Rightarrow \quad zx > zy && \text{(exercise: } -(-a) = a\text{).}
\end{aligned}$$

3. Suppose that $x \neq 0$. By trichotomy, either $x > 0$ or $x < 0$. We consider these cases separately. If $x > 0$, then use multiplicative translation:

$$x > 0 \quad \Rightarrow \quad x \cdot x > x \cdot 0 \quad \Rightarrow \quad x^2 > 0.$$

If $x < 0$, use part 2:

$$x < 0 \quad \Rightarrow \quad x \cdot x > x \cdot 0 \quad \Rightarrow \quad x^2 > 0.$$

4. By part 3, we have $1 = 1^2 > 0$. Then applying part 1,

$$0 < 1 \quad \Rightarrow \quad -1 < -0 \quad \Rightarrow \quad -1 < 0.$$

5. Left as an exercise.

□

**Exercise.** Can the field $\mathbb{Z}/5\mathbb{Z}$ be ordered, i.e., can a relation $<$ on $\mathbb{Z}/5\mathbb{Z}$ be defined that satisfies the order axioms?

**Absolute value.** Here we define the absolute value function for a ordered field, and prove some of its basic properties. We then prove one of the most important theorems in analysis—the triangle inequality. It will soon become one of our main tools.

**Definition.** Let $F$ be an ordered field. The *absolute value* of $x \in F$ is

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

**Proposition 2.** For all $x$ in an ordered field $F$,

$$-|x| \leq x \leq |x|.$$

*Proof.* There are three cases (by trichotomy):

$x = 0$: In this case $-|x| = x = |x| = 0$, and the result holds.

$x > 0$: In this case $|x| = x$. Hence, $x \leq x = |x|$, which is the right half of the result. For the left half, note that $|x| = x > 0$, implies $-|x| < 0$. Then, since $-|x| < 0$ and $0 < x$, by transitivity $-|x| \leq x$.

$x < 0$: In this case, $|x| = -x > 0$, which implies $x < 0 < |x|$. By transitivity, $x \leq |x|$. On the other hand $|x| = -x$ implies $-|x| = x$. In particular, $-|x| \leq x$. □

**Proposition 3.** If $x$ and $a$ are elements of an ordered field $F$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.

*Proof.* This proof is similar to that of Proposition 1 and is left as an exercise for the interested reader. □

As mentioned above, the following result will be very important in the sequel. It is worth reading its short proof.

**Theorem.** (Triangle inequality.) If $x$ and $y$ are elements of an ordered field, then

$$|x + y| \leq |x| + |y|.$$

*Proof.* By Proposition 2, we have

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

By Proposition 1, part 5, we can add these inequalities to get

$$-|x| - |y| \leq x + y \leq |x| + |y|,$$

and, hence,

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

The result now follows by letting $a = |x| + |y|$ in Proposition 3. □

**Example.** Here are several instances of the triangle inequality in the case of $\mathbb{Q}$ or $\mathbb{R}$:

$$5 = |2 + 3| \leq |2| + |3| = 5$$

$$1 = |-2 + 3| \leq |-2| + |3| = 5$$

$$5 = |-2 - 3| \leq |-2| + |-3| = 5.$$

When, in general, does *equality* hold in the triangle inequality?