Math 112 lecture for Friday, Week 5

Extrema

(Supplemental reading: Sections 2.6 and 2.7 in Swanson.)

Before getting to the main propositions for today, we provide some templates for proving statements involving bounds for sets.

Proof template. Define a subset $S \subset \mathbb{R}$ by blah, blah, blah, and let B be the real number blah. Then B is an upper bound for S.

Proof. Let $s \in S$. Then blah, blah, blah. It follows that $s \leq B$. □

Proof template. Define a subset $S \subset F$ by blah, blah, blah, and let B be the real number blah. Then $B = \sup S$.

Proof. We first show that B is an upper bound for S. Let $s \in S$. Then blah, blah, blah. It follows $s \leq B$. Next we show that B is a least upper bound for S. Suppose that B' is an upper bound for S. Then blah, blah, blah. It follows that $B \leq B'$. □

An alternative, to the preceding template, which is sometimes easy to use:

Proof template. Define a subset $S \subset \mathbb{R}$ by blah, blah, blah, and let B be the real number blah. Then $B = \sup S$.

Proof. We first show that B is an upper bound for S. Let $s \in S$. Then blah, blah, blah. It follows $s \leq B$. Next we show that B is a least upper bound for S. Suppose $B' < B$. Then B' is not an upper bound for S since blah, blah, blah. □

The templates for lower bounds and infima are similar.

Example. Claim: Let $S = (-\infty, 1)$. Then $\sup(S) = 1$.

Proof. We first show that 1 is an upper bound for S. This follows immediately from the definition of $S = (-\infty, 1) := \{x \in \mathbb{R} : x < 1\}$. So if $s \in S$ then $s < 1$. Next, we show that 1 is a least upper bound for S. Suppose $x < 1$. Then $(x + 1)/2$, the midpoint between x and 1, is an element of S and is greater than x, it follows that x is not an upper bound for S. In sum, 1 is an upper bound for S and anything smaller then 1 is not an upper bound. So $1 = \sup(S)$. □

We move on now to our main results for this lecture. The last two are especially interesting and important for what is to come.

Suppose F is an ordered field and $S \subseteq F$. Define the subset $-S \subseteq F$ by

$$-S := \{-s : s \in S\} .$$

For instance, if $S = (2,5) \subset \mathbb{R}$, then $-S = (-5,-2)$. By our first proposition, below, $\inf(-S) = -5 = -\sup(S)$. Note that, in general, $-(-S) = S$.

Proposition 1. Using the notation from above,

(a) if $\inf(-S)$ exists, then $\sup(S)$ exists and $\sup(S) = -\inf(-S)$;

(b) if $\sup(S)$ exists, then $\inf(-S)$ exists and $\inf(-S) = -\sup(S)$.

Proof. We will just prove the first part, the second being similar. Suppose that $\inf(-S)$ exists, and for ease of notation, let $t = \inf(-S)$. We must show that $\sup(S)$ exists and $\sup(S) = -t$. We first show that $-t$ is an upper bound for S. Take $s \in S$. Then $-s \in -S$, and so it follows that $t \leq -s$ (by definition of $\inf(-S)$). It follows that $-t \geq s$. Hence, $-t$ is an upper bound for S. Next, suppose that x is any upper bound for S. We first claim that $-x$ is a lower bound for $-S$: given $y \in -S$, we have $-y \in S$, and hence, $x \geq -y$ (since x is an upper bound for S). It follows that $-x \leq y$. We have shown that $-x$ is a lower bound for $-S$. Since $t = \inf(-S)$ is the greatest lower bound for $-S$, we have $-x \leq t$. From this, it follows that $x \geq -t$. We have shown that $-t$ is the least upper bound for S, i.e., $-t = -\inf(-S) = \sup(S)$. $\square$

For the next proposition, recall that an ordered field is complete if each of its nonempty subsets that is bounded above has a least upper bound, i.e., a supremum.

Proposition 2. Let F be an ordered field, and suppose that every nonempty subset of F that is bounded below has an infimum. Then F is complete.

Proof. Let $\emptyset \neq S \subseteq F$, and suppose S is bounded above, say by $B \in F$. We would like to show that S has a supremum. First, note that $-B$ is a lower bound for $-S$: if $x \in -S$, then $-x \in S$, and hence, $B \geq -x$. It follows that $-B \leq x$. Next, since $-S$ is bounded below, by hypothesis, $-S$ has an infimum $\inf(-S)$. Then by Proposition 1, we see that $\sup(S)$ exists (and is equal to $-\inf(-S)$). We have shown that every nonempty subset of F that is bounded above has a least upper bound, i.e., F is complete. $\square$

The next result shows that there are **no infinitely small positive elements** in any ordered field!

Proposition 3 (No infinitesimals.) Let F be an ordered field. Let $x \in F$ and suppose that

$$0 \leq x \leq y$$

for all $y \in F$ with $y > 0$. Then $x = 0$. In other words, the only nonnegative element of F that is less than or equal to all the positive elements of F is 0. No positive element of F can be less than or equal to all the positive elements of F.

Proof. Since $x \geq 0$, by trichotomy there are two possibilities: $x > 0$ or $x = 0$. For sake of contradiction, suppose that $x > 0$. Then let $y := x/2$. Below, using the order axioms, we will prove that $y < x$ and $y > 0$ in contradiction to the hypotheses concerning x, i.e., that $0 \leq x \leq y$ for $y > 0$. The only possibility then left is that $x = 0$, as desired.

The reasoning we just used for y is clearly in agreement with our experience with the rational or real numbers. However, we are working in an arbitrary ordered field. So we now check that, in general, if $x > 0$ and $y = x/2$, then $y < x$ and $y > 0$.

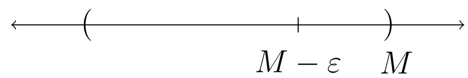
We have already shown that since F is an ordered field, $1 > 0$. By additive translation, adding 1 to both sides of the inequality, we get $2 := 1 + 1 > 1$. By transitivity, $2 > 1 > 0$ implies $2 > 0$. Therefore, $2 \neq 0$ (again by trichotomy). The point of all of the above is that $2 \neq 0$, and hence has a multiplicative inverse, which we naturally denote by $1/2$. Why is $1/2 > 0$, necessarily? Answer: by trichotomy, either $1/2 = 0$, $1/2 < 0$, or $1/2 > 0$. We rule out the first two possibilities. If $1/2 = 0$, then $1 = 2 \cdot (1/2) = 2 \cdot 0 = 0$. But $1 \neq 0$ in a field. If $1/2 < 0$, then multiplying through $2 > 0$ by $1/2$ would yield $1 < 0$, which we saw in an earlier lecture is impossible.

Continuing: since $2 > 1$, multiplicative translation by $1/2$ gives $1 > 1/2$. Then multiplicative translation by $x > 0$ gives $x > x/2$, i.e., $x > y$. Finally, to see that $y > 0$, start with $x > 0$. Multiplicative translation by $1/2 > 0$ yields $x/2 > (1/2) \cdot 0 = 0$, i.e., $y > 0$. □

The next proposition will be very useful later on. It says that if a set has a supremum, then even if the supremum is not an element of the set, then if can be approximated arbitrarily closely with an element of the set. For example, consider the interval $S := (0, 1) \subset \mathbb{R}$. Then $\sup(S) = 1 \notin S$. Can we approximate $\sup(S)$ to within a tolerance of $\varepsilon := 0.001$ by an element in the set? Sure: take $x := 1 - 0.0001 = 0.9999$, for instance. Then $x \in S$ and is within 0.001 of the supremum, 1.

Proposition 4. Let S be a subset of an ordered field F, and suppose that $M := \sup S$ exists. Given $\varepsilon \in F$ with $\varepsilon > 0$, there exists $s \in S$ such that $M - s < \varepsilon$.

Proof. To help with understanding this proof, here is a picture for the case where S is an interval in $F = \mathbb{R}$:



Let $s \in S$. Note that the statement $M - s < \varepsilon$ is equivalent to the statement $M - \varepsilon < s$. Further, if either of these statements holds, then since $M = \sup(S)$ and $s \in S$ it follows that, in addition to $M - \varepsilon < s$, we have $s \leq M$, i.e., s is in the interval $(M - \varepsilon, M]$. Thus, we are trying to argue that for any $\varepsilon > 0$, we can find an s in S that's between $M - \varepsilon$ and M. Of course, if $M \in S$, we could take $M = s$, but it's not always the case that a set contains its supremum.

The proof starts here. By the additive translation order axiom, we can add $M - \varepsilon$ to both sides of the inequality $0 < \varepsilon$ to get

$$(M - \varepsilon) + 0 < (M - \varepsilon) + \varepsilon.$$

Using associativity and addition and the definition of the additive inverse $-\varepsilon$, we see that

$$M - \varepsilon < M.$$

Since M is the least upper bound of S, we know that if M' is an upper bound of S, then $M \leq M'$. In particular, $M - \varepsilon$ cannot be an upper bound of S because $M - \varepsilon < M$. (In other words, since $M - \varepsilon$ is strictly smaller than the least upper bound, it cannot be an upper bound.) The fact that $M - \varepsilon$ is not an upper bound of S, means there exists some $s \in S$ such that $M - \varepsilon < s$. We now add $\varepsilon - s$ to both sides of this inequality to obtain the desired inequality:

$$M - s = (M - \varepsilon) + (\varepsilon - s) < s + (\varepsilon - s) = \varepsilon.$$

To summarize: since $M - \varepsilon$ is strictly smaller than the least upper bound of S, it cannot be an upper bound for S. This means that there exists some $s \in S$ such that $M - \varepsilon < s$, and the result follows. □

A similar proposition holds for infima.