Math 112 addendum to lecture for Friday, Week 4

Here we provide several more examples of implications of the field axioms. Each of the results is a basic property of $\mathbb{Q}$ and of $\mathbb{R}$. We would like to see how they follow solely from the field axioms (and, thus, hold for any field).

First, the field axioms dictate the existence of an additive identity, but they nowhere stipulate that the additive identity is *unique*. Could there be two distinct elements of a field, say 0_1 and 0_2, such that both $0_1 + x = x$ and $0_2 + x = x$ for all x in the field? It turns out that the answer is "no":

Proposition. Let F be a field. Then F has a unique additive identity.

Proof. Suppose that 0_1 and 0_2 are both additive identities for F. This means that

$$0_1 + x = x \quad \text{and} \quad 0_2 + x = x$$

for all $x \in F$. Then,

$$\begin{aligned} 0_1 &= 0_1 + 0_2 && \text{(since } 0_2 \text{ is an additive identity)} \\ &= 0_2 && \text{(since } 0_1 \text{ is an additive identity).} \end{aligned}$$

Thus, $0_1 = 0_2$. □

A similar argument proves that a field has a unique multiplicative identity.

Proposition. (Cancellation law for addition.) Let F be a field, and let $x, y, z \in F$. Then

$$x + y = x + z \quad \Rightarrow \quad y = z.$$

Proof. Since F is a field, x has an additive inverse. Therefore,

$$\begin{aligned} x + y = x + z \quad &\Rightarrow \quad -x + (x + y) = -x + (x + z) && \text{(a property of } =\text{)} \\ &\Rightarrow \quad (-x + x) + y = (-x + x) + z && \text{(associativity of addition)} \\ &\Rightarrow \quad 0 + y = 0 + z && \text{(definition of } -x\text{)} \\ &\Rightarrow \quad y = z && \text{(0 is the additive identity).} \end{aligned}$$

□

Proposition. Let F be a field, and let $x, y \in F$. Then $xy = 0$ if and only if $x = 0$ or $y = 0$.

Proof. (⇒) Suppose that $xy = 0$ but $x \neq 0$. Since F is a field, x has a multiplicative inverse x^{-1}. Then

$$\begin{aligned}
xy = 0 \quad &\Rightarrow \quad x^{-1}(xy) = x^{-1} \cdot 0 && \text{(property of =)} \\
&\Rightarrow \quad x^{-1}(xy) = 0 && \text{(previous proposition)} \\
&\Rightarrow \quad (x^{-1}x)y = 0 && \text{(associativity of } \cdot) \\
&\Rightarrow \quad 1 \cdot y = 0 && \text{(definition of } x^{-1}) \\
&\Rightarrow \quad y = 0 && \text{(1 is the mult. identity).}
\end{aligned}$$

Similarly, if $y \neq 0$, then $x = 0$.

(⇐) Conversely, if either $x = 0$ or $y = 0$, then the previous proposition implies $xy = 0$. □

Example. In contrast to the proposition we just proved, note that in $\mathbb{Z}/6\mathbb{Z}$, we have $[2] \neq [0]$ and $[3] \neq [0]$, but

$$[2][3] = [0].$$

Thus, $\mathbb{Z}/6\mathbb{Z}$ is not a field.

Proposition. Let F be a field, and let $x \in F$. Then

$$(-1)x = -x.$$

Proof. (Note that there is something to prove here! On the left we have the product of the additive inverse of 1 with x, and on the right we have the additive inverse of x. To show they are equal, we need to show $(-1)x$ is the additive inverse of x, i.e., if we add it to x, we get 0.)

Compute:

$$\begin{aligned}
x + (-1)x &= 1 \cdot x + (-1)x && \text{(1 is the multiplicative identity)} \\
&= (1 + (-1))x && \text{(distributivity)} \\
&= 0 \cdot x && \text{(definition of } -1) \\
&= 0 && \text{(proved earlier).}
\end{aligned}$$

□