Math 112 lecture for Wednesday, Week 3

INJECTIVITY AND SURJECTIVITY

Recall from last time:

Definition. Let $f\colon A \to B$ be a function. Then

1. f is *injective* or *one-to-one* if for all $a, a' \in A$:

$$f(a) = f(a') \quad \Rightarrow \quad a = a'.$$

2. f is *surjective* or *onto* if

$$\mathrm{im}(f) = B.$$

3. f is *bijective* if it is injective and surjective (one-to-one and onto).

Exercise. Are the following functions injective? surjective? bijective?

1.

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

2.

$$\begin{aligned} g\colon \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

3.

$$\begin{aligned} h\colon [0, 2\pi] &\to \mathbb{R} \\ x &\mapsto \cos(x). \end{aligned}$$

4.

$$\begin{aligned} k\colon [0, 2\pi] &\to [-1, 1] \\ x &\mapsto \cos(x). \end{aligned}$$

Definition. Let $f\colon A \to B$ be a function between sets A and B, and let $C \subseteq A$. The *image of* C *under* f is the subset of B

$$f(C) := \{f(c) : c \in C\} \subseteq B.$$

Let $D \subseteq B$. The *inverse image of* D *under* f is the subset of A

$$f^{-1}(D) := \{a \in A : f(a) \in D\}.$$

Examples.

1. Let

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2. \end{aligned}$$

Then

$$f([0,2]) = [0,4],$$

and

$$f^{-1}([-1,3]) = [-\sqrt{3}, \sqrt{3}].$$

2. Let

$$\begin{aligned} f\colon \mathbb{Z} &\to \mathbb{Z}/4\mathbb{Z} \\ a &\mapsto [a]. \end{aligned}$$

Then

$$f(\{0,2,4,6,8,\dots\}) = \{[0],[2]\}.$$

and

$$f^{-1}([3]) = \{\dots,-9,-5,-1,3,7,11,15,\dots\}.$$

Proposition. If $f\colon A \to B$ and $C \subseteq A$, then

$$C \subseteq f^{-1}(f(C)).$$

Proof. Let $c \in C$. Then $f(c) \in f(C)$. Hence $c \in f^{-1}(f(C))$. $\square$

Remark. The proposition we just proved is no longer true if $\subseteq$ is replaced by $=$. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $C = [0, \infty)$. Then

$$f^{-1}(f([0,\infty))) = (-\infty,\infty) = \mathbb{R} \neq [0,\infty).$$

Proposition. Let $f\colon A \to B$, and let $X, Y \subseteq A$. Then

$$f(X \cap Y) \subseteq f(X) \cap f(Y).$$

Proof. Let $z \in f(X \cap Y)$. Then $z = f(a)$ for some $a \in X \cap Y$. Since $a \in X \cap Y$, it follows that $a \in X$, and hence, $z = f(a) \in f(X)$. Similarly, since $a \in X \cap Y$, it follows that $a \in Y$, and hence $z = f(a) \in f(Y)$. We've shown $z \in f(X)$ and $z \in f(Y)$, and therefore, $z \in f(X) \cap f(Y)$. $\square$

Challenge. Find an example of a function $f\colon A \to B$ and subsets X, Y of A such that

$$f(X \cap Y) \neq f(X) \cap f(Y).$$

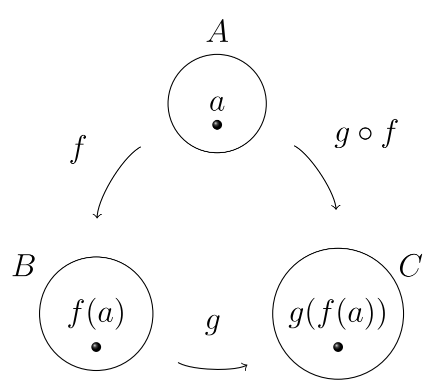
Thus, we don't necessarily have equality in the previous proposition.

COMPOSITION OF FUNCTIONS

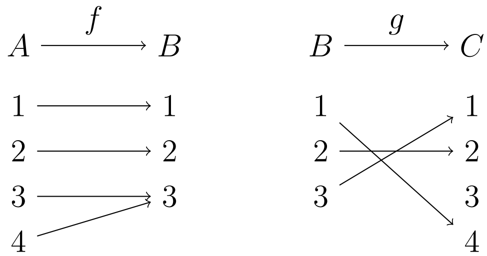
Definition. Let $f\colon A \to B$ and $g\colon B \to C$ be functions. The *composition of f and g* is the function

$$g \circ f\colon A \to C$$

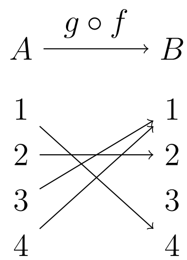
defined by $(g \circ f)(a) := g(f(a))$.



Example. Consider the two functions pictured below:



The composition $g \circ f$ is then:



Note that $f \circ g$ would not make sense since the codomain of g is not the domain of f.

Proposition. The composition of surjective functions is surjective.

Proof. Let $f\colon A \to B$ and $g\colon B \to C$ be surjective functions. We want to show that $g \circ f\colon A \to C$ is surjective. Let $c \in C$. since g is surjective, there exists $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Then,

$$(g \circ f)(a) := g(f(a)) = g(b) = c.$$

We have shown that every element of C is in the image of $g \circ f$, i.e., $g \circ f$ is surjective. Therefore, $g \circ f$ is surjective. □

Proposition. The composition of injective functions is injective.

Proof. Let $f\colon A \to B$ and $g\colon B \to C$ be injective functions. We want to show that $g \circ f$ is injective. So let $a, a' \in A$ and suppose that $(g \circ f)(a) = (g \circ f)(a')$. In other words, $g(f(a)) = g(f(a'))$. Since g is injective, $f(a) = f(a')$. Then, since f is injective $a = a'$. Thus, $g \circ f$ is injective. □

Corollary. The composition of bijective functions is bijective.

Proof. This is an immediate corollary of the preceding two propositions. □

Inverse functions

Definition. Suppose $f\colon A \to B$ is a bijective function. Then the *inverse of* f is the function $g\colon B \to A$ defined by

$$g(b) = a \quad \text{if} \quad f(a) = b.$$

This function g denoted by f^{-1}. Hence, $f^{-1}(b) = a$ if and only if $f(a) = b$.

Remark. Let $f\colon A \to B$. Earlier, for every subset $C \subset B$, we defined $f^{-1}(C) = \{a \in A : f(a) \in C\}$. If f is bijective, then we have two different but closely related meanings for f^{-1}. If $b \in B$, then $f^{-1}(\{b\})$ is a subset of A consisting of a single element, say a. So $f^{-1}(\{b\}) = \{a\}$. That's the earlier meaning of f^{-1}, defined for subsets of B. For the inverse of f, just defined, we have $f^{-1}(b) = a$.

Exercise. For each of the following functions, state why there is no inverse, or describe the inverse function.

1.

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

2.

$$\begin{aligned} g\colon \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

3.

$$\begin{aligned} h\colon \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto |x|. \end{aligned}$$

Definition. The *identity function* on a set S is the function

$$\begin{aligned} \mathrm{id}_S\colon S &\to S \\ x &\mapsto x. \end{aligned}$$

One may show the following:

1. The function $f\colon A \to B$ has an inverse if and only if there exists a function $g\colon B \to A$ such that
$$f \circ g = \mathrm{id}_B \quad \text{and} \quad g \circ f = \mathrm{id}_A.$$
In this case, $g = f^{-1}$.

2. If g is the inverse of f, then f is the inverse of g.

3. If $f\colon A \to B$ and $g\colon B \to C$ have inverse, then so does $g \circ f\colon A \to C$, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise. Consider the functions $f(x) = x + 1$ and $g(x) = 3x$, both with domain and codomain $\mathbb{R}$. Compute the following: (i) $g \circ f$, (ii) $(g \circ f)^{-1}$, (iii) f^{-1}, (iv) g^{-1}, and (v) $f^{-1} \circ g^{-1}$. You should get $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.