Math 112 lecture for Monday, Week 3

(Supplemental reading: Section 2.4 in Swanson.)

Rule for proof-writing. When proving something is *not* the case, always give as simple a counter-example as possible. This is the opposite of trying to prove something *is* the case, in which you must be as general as possible (i.e., don't give a "proof by example"). For instance, if I want to disprove the statement "There are no prime numbers that are 1 modulo 4", I can say: "The statement is not true. For example, 5 is prime and $5 = 1 \bmod 4$."

FUNCTIONS

What is a function? To work up to the definition, think about some function that you know. For instance, consider the function $f(x) = x^2$ from calculus. The *graph* of f is the set

$$\left\{(x, x^2) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}.$$

Note that f is completely determined by its graph. To see this point clearly, note that if I told you that I am thinking of a function whose graph is

$$\left\{(x, 2\cos(x) + 3) \in \mathbb{R}^2 : x \in \mathbb{R}\right\},$$

you would be able to tell me that the function is $g(x) = 2\cos(x) + 3$.

Next, note that the graph of a function is actually a relation between sets. For instance, in the above two examples, it's a relation between $\mathbb{R}$ and itself, i.e., a relation on $\mathbb{R}$. Is every relation on $\mathbb{R}$ a function? The answer is "no" since a function must be single-valued. For instance, if $h(1) = 4$, we cannot also have $h(1) = 7$. So we can't have both $(1, 4)$ and $(1, 7)$ in the graph of h.

Definition. Let A and B be sets. A *function* f with *domain* A and *codomain* B, denoted $f\colon A \to B$ is a relation

$$R_f \subseteq A \times B$$

such that:

1. For all $a \in A$, there exists $b \in B$ such that $(a, b) \in R_f$.
2. If (a, b) and (a, b') are in R_f, then $b = b'$.

If $(a, b) \in R_f$, then we write $f(a) = b$.

Remarks. Part 1 says that a function needs to be defined at each point in its domain. Part 2 says that a function is single-valued. Note that R_f, which we are using to define f, is exactly the graph of f.

Example. Let $S = \{1, 2, 3\}$ and $T = \{0, 1\}$. Define a function $f\colon S \to T$ by letting $f(1) = 0$ and $f(2) = f(3) = 1$.

1. What is the domain of f?

2. What is the codomain of f?

3. What is the relation R_f defining f?

(See the footnote[1] for the solution.)

Definition. Let $f\colon A \to B$ be a function. The *image* or *range* of a function is the subset of B

$$\mathrm{im}(f) := \{f(a) \in B : a \in A\}\,.$$

Examples.

1. Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3\}$, and define a function $f\colon A \to B$ by $f(a) = f(b) = 1$ and $f(c) = 2$:

$$\begin{tikzcd}
A \arrow[r] & B \\
a \arrow[r] & 1 \\
b \arrow[ur] & 2 \\
c \arrow[ur] & 3
\end{tikzcd}$$

   We have

$$\begin{aligned}
\text{domain of } f\colon &\quad A = \{a, b, c\} \\
\text{codomain of } f\colon &\quad B = \{1, 2, 3\} \\
\text{image of } f\colon &\quad \{1, 2\}\,.
\end{aligned}$$

   Note: The words "image" and "range" mean the same thing. As is evident in the above example, the image is a subset of the codomain, but not necessarily equal to the codomain.

[1] The domain is $S = \{1, 2, 3\}$, the codomain is $T = \{0, 1\}$, and the relation is $R_f = \{(1, 0), (2, 1), (3, 1)\}$

2. The image of the function

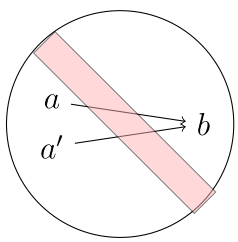
$$\begin{aligned} g\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x-3| \end{aligned}$$

is $\mathbb{R}_{\geq 0}$, the set of nonnegative real numbers. The domain and codomain of g are both $\mathbb{R}$.

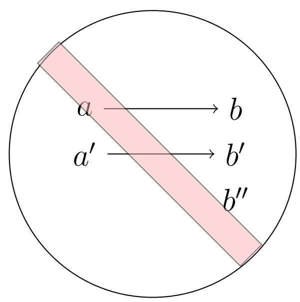
Definition. Let $f\colon A \to B$ be a function. Then

1. f is *injective* or *one-to-one* if for all $a, a' \in A$:
$$f(a) = f(a') \quad \Longrightarrow \quad a = a'.$$
2. f is *surjective* or *onto* if
$$\operatorname{im}(f) = B.$$
3. f is *bijective* if it is injective and surjective (one-to-one and onto).

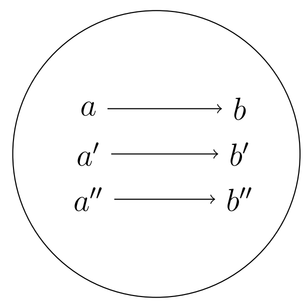
"International symbols":



not injective



injective
not surjective



bijective

Proof templates. Here are templates for proving a function is injective, surjective, or bijective:

Proposition. The function $f\colon A \to B$ is injective.

Proof. Let $x, y \in A$, and suppose that $f(x) = f(y)$. Then blah, blah, blah. It follows that $x = y$. Hence, f is injective. $\square$

Proposition. The function $f\colon A \to B$ is surjective.

Proof. Let $b \in B$. Then blah, blah, blah. Thus, there exists $a \in A$ such that $f(a) = b$. Hence, f is surjective. $\square$

Proposition. The function $f\colon A \to B$ is bijective.

Proof. We first show that f is injective. [Follow the template above to show injectivity.] Next, we show f is surjective. [Follow the template above to show surjectivity.] $\square$

Later, we will see alternative proofs of these propositions using (left and right) inverse functions.

Examples.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. Then f is bijective.

   Proof. To see f is injective, let $x, y \in \mathbb{R}$ and suppose that $f(x) = f(y)$. Since $f(x) = f(y)$, we have that $2x = 2y$. Dividing by 2, we see that $x = y$. Hence, f is injective.

   To see that f is surjective, let $z \in \mathbb{R}$ (in the codomain). Then, in the domain, we have $z/2 \in \mathbb{R}$, and $f(z/2) = 2(z/2) = z$. Hence, f is surjective.

   Since f is injective and surjective, it is bijective. $\square$

2. The function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

   is neither injective nor surjective. It's not injective since, for instance, $f(1) = f(-1)$, and it's not surjective since its image is $\mathbb{R}_{\geq 0}$, which is not equal to the codomain, $\mathbb{R}$. For instance, -1 is not in the image of f.

Challenge. Can you find $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that the function $A \to B$ defined by $x \mapsto x^2$ is injective but not surjective? surjective but not injective? bijective?