Math 112 lecture for Wednesday, Week 2

Relations and equivalence relations

(Supplemental reading: Section 2.3 in Swanson.)

Relations.

Definition. A *relation* between sets A and B is a subset R of their Cartesian product:

$$R \subseteq A \times B.$$

If $(a, b) \in R$, we may write aRb. If $A = B$, we say R is a *relation on* A.

Examples.

(a) First, we consider a toy example that does not have much meaning. Let $A = \{\checkmark, \star\}$ and $B = \{1, 2, 3\}$, and $R = \{(\checkmark, 2), (\checkmark, 3), (\star, 2)\}$. Then $\checkmark R\, 2$, $\checkmark R\, 3$, and $\star R\, 2$.

(b) For a more serious example we see how the "less than or equal" relation on the integers, $\mathbb{Z}$, can be thought of as a relation in the technical sense we have just introduced. We just take

$$R = \{(a, b) : a, b \in \mathbb{Z} \text{ and } a \leq b\}.$$

So in this case, we have aRb if and only if $a \leq b$.

A relation R on a set S, i.e., $R \subseteq S \times S$, is an *equivalence relation on* S if for all $x, y, z \in S$:

1. xRx (the relation is reflexive)
2. If xRy, then yRx (the relation is symmetric)
3. If xRy and yRz, then xRz (the relation is transitive).

If R is an equivalence relation on S, we usually write $a \sim b$ instead of aRb. So re-writing the axioms for an equivalence relation, we require for all $x, y, z \in S$:

1. $x \sim x$ (the relation is reflexive)
2. If $x \sim y$, then $y \sim x$ (the relation is symmetric)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$ (the relation is transitive).

Examples.

1. Is $\leq$ an equivalence relation on $\mathbb{Z}$? It's reflexive: $a \leq a$ for all $a \in \mathbb{Z}$. It's transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$. However, it's not symmetric: $a \leq b$ does not necessarily imply $b \leq a$. For example $0 < 1$ but $1 \not< 0$. So $\leq$ is a relation on $\mathbb{Z}$, but it is not an equivalence relation.

2. The relation $=$ is an equivalence relation on any set.

The integers modulo n. We now describe a whole class of equivalence relations on the set of integers, $\mathbb{Z}$—one for each $n \in \mathbb{Z}$. First, fix your favorite integer $n \in \mathbb{Z}$. Then for $a, b \in \mathbb{Z}$ we will say $a \sim b$ if a and b differ by a multiple of n: i.e., if

$$a - b = kn$$

for some $k \in \mathbb{Z}$.

We will prove we get an equivalence relation in the next lecture. For now let's look at some examples. The case of $n = 2$ says $a, b \in \mathbb{Z}$ are equivalent if a and b differ by a multiple of 2. So we have the following equivalences

$$\cdots \sim -4 \sim -2 \sim 0 \sim 2 \sim 4 \sim \cdots .$$

and we have

$$\cdots \sim -3 \sim -1 \sim 1 \sim 3 \sim 5 \sim \cdots .$$

So under this equivalence relation, the even integers are all equivalent to each other, and the odd integers are equivalent to each other. No even integer is equivalent to an odd integer. We say there are two "equivalence classes".

For another example, consider the relation in the case $n = 3$: two integers are equivalent if they differ by a multiple of 3. In that case we have three equivalence classes:

$$\cdots \sim -6 \sim -3 \sim 0 \sim 3 \sim 6 \sim \cdots \tag{1}$$
$$\cdots \sim -5 \sim -2 \sim 1 \sim 4 \sim 7 \sim \cdots \tag{2}$$
$$\cdots \sim -4 \sim -1 \sim 2 \sim 5 \sim 8 \sim \cdots . \tag{3}$$

Equivalence relations and partitions. A *partition* of a set S is a collection of nonempty subsets S_k satisfying:

(a) the S_k are pair-wise disjoint: for all i, j, we have $S_i \cap S_j = \emptyset$, and

(b) the union of the S_k is S: we have $\cup_k S_k = S$.

Example. Let $S = \{1, 2, 3, 4, 5, 6\}$. The following sets form a partition of S:

$$S_1 := \{2, 4\}, S_2 := \{1, 3, 5\}, S_3 := \{6\}.$$

No two of these sets share an element, and their union is all of S.

Fact 1: For every partition of a set S, there is an equivalence relation on S whose equivalence classes are the subsets in the partition. The equivalence relation is defined by requiring the elements in each subset of the partition to be related to each other.

Continuing with the previous example, we are looking for a equivalence relation on $S = \{1, 2, 3, 4, 5, 6\}$ whose equivalence classes are S_1, S_2 and S_3. First considering $S_1 = \{2, 4\}$, we require $2 \sim 2$, $4 \sim 4$, $2 \sim 4$, and $4 \sim 2$. The other subsets, S_2 and S_3 are handled similarly, producing the required equivalence relation on S.

We have a converse to the above fact:

Fact 2: Given an equivalence relation on a set S, its set of equivalence classes partitions S.

As an example, consider the integers modulo 3. In (1), above, we saw that we get three equivalence classes: one containing 0, one containing 1, and one containing 2. One may check that these equivalence classes partition the set $\mathbb{Z}$: every integer is in exactly one of these classes.

Facts 1 and 2 show that equivalence relations and partitions are essentially the same thing. The interested reader could attempt to give a formal proof.