Math 112 lecture for Monday, Week 2

More sets, Cartesian products

(Supplemental reading: Sections 2.1 and 2.2 in Swanson.)

Template.

Proposition. If [some hypotheses go here], then

$$A = B.$$

Proof. Let $a \in A$. Then [use hypotheses, definitions, calculations here]. Therefore, $a \in B$. Hence, $A \subseteq B$.

Conversely, let $b \in B$. Then [use hypothesese, definitions, calculations here]. Therefore, $b \in A$. Hence, $B \subseteq A$, too. Therefore, $A = B$. □

An example:

Proposition. Let A and B be sets, and let $C := A \cup B$. Suppose $A \cap B = \emptyset$. Then

$$A = C \setminus B.$$

Proof. Let $a \in A$. Then $a \in C = A \cup B$. Since $A \cap B = \emptyset$ and $a \in A$, it follows that $a \notin B$. In sum, $a \in C$ and $a \notin B$. Therefore, $a \in C \setminus B$. Thus $A \subseteq C \setminus B$.

Conversely, let $x \in C \setminus B$. This means that $x \in C$ and $x \notin B$. But $x \in C = A \cup B$, means that $x \in A$ or $x \in B$. Since $x \notin B$, it follows that $x \in A$. Therefore, $C \setminus B \subseteq A$. □

Indexed unions and intersections. Let I be a set, and suppose that for each $i \in I$, you are given a set A_i. Then by definition,

$$\cup_{i \in I} A_i := \{x : x \in A_i \text{ for some } i \in I\}$$
$$\cap_{i \in I} A_i := \{x : x \in A_i \text{ for all } i \in I\}.$$

If $I = \mathbb{N}$, we might write $\cup_{i=1}^{\infty} A_i$ in place of $\cup_{i \in \mathbb{N}^+} A_i$, and similarly for intersections. In that case, your can think of these operations as follows:

$$\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots$$

$$\cap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots$$

Examples.

1. For each $n \in \mathbb{N}^+$, let $A_n = [0, 1/n)$, an interval in $\mathbb{R}$. Then
   (a) $\cup_{n\in\mathbb{N}^+} A_n = [0, 1)$.
   (b) $\cap_{n\in\mathbb{N}^+} A_n = \{0\}$.
2. $\cup_{r\in\mathbb{R}} \{r\} = \mathbb{R}$.

We will prove $\cap_{n\in\mathbb{N}^+} A_n = \{0\}$. We need to show these two sets are equal, so we show inclusions in both directions. I find that when possible, it helps to first get an intuitive grasp by writing out the indexed intersection long-hand:

$$\cap_{n\in\mathbb{N}^+} A_n = \cap_{n\in\mathbb{N}^+}[0, 1/n) = [0, 1) \cap [0, 1/2) \cap [0, 1/3) \cap \cdots$$

Each successive interval is contained in the preceding one. So the intersection is getting smaller as we go out in the chain. Now for a formal proof:

Proof. Let $x \in \cap_{n\in\mathbb{N}^+} A_n$. Then $x \in A_n = [0, 1/n)$ for all $n \in \mathbb{N}$. Thus,

$$0 \leq x < \frac{1}{n}.$$

This means that $x = 0$ (which we won't prove here). Therefore, $x \in \{0\}$. We have shown the inclusion

$$\cap_{n\in\mathbb{N}^+} A_n \subseteq \{0\}.$$

For the opposite inclusion: there is only one element of $\{0\}$, namely 0, and $0 \in [0, 1/n)$ for $n = 1, 2, \ldots$ Therefore, $0 \in \cap_{n\in\mathbb{N}^+} A_n$, and hence

$$\{0\} \subseteq \cap_{n\in\mathbb{N}^+} A_n.$$

Having shown both inclusions, we know the sets are equal. $\square$

CARTESIAN PRODUCTS

Definition. The *Cartesian product* of sets A and B is

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$

By (a, b) we mean an "ordered pair". (Formally, we could define $(a, b) := \{a, \{a, b\}\}$.) For example, $(1, 2) \neq (2, 1)$, whereas $\{1, 2\} = \{2, 1\}$. By definition,

$$(a, b) = (a', b') \quad \text{exactly when} \quad a = a \text{ and } b = b'.$$

Examples.

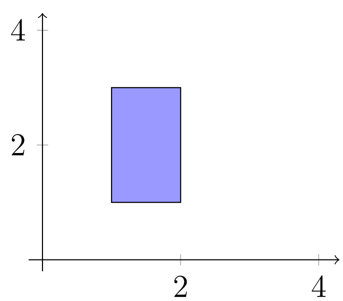
1. Let $A = \{\checkmark, \star\}$ and $B = \{1, 2, 3\}$. Then

$$A \times B = \{(\checkmark, 1), (\checkmark, 2), (\checkmark, 3), (\star, 1), (\star, 2), (\star, 3)\}\,.$$

2. Let $A = [1, 2] \subset \mathbb{R}$ and $B = [1, 3] \subset \mathbb{R}$. Then

$$A \times B = \{(a, b) : 1 \leq a \leq 2 \text{ and } 1 \leq b \leq 3\}\,.$$

This is a rectangle in the plane $\mathbb{R}^2$:



$[1, 2] \times [1, 3]$.

3. Let $A = B = \mathbb{R}$. Then $A \times B = \mathbb{R}^2$, the ordinary real plane.

Given sets A, B, C, we can define

$$A \times B \times C := \{(a, b, c) : a \in A, b \in B, \text{ and } c \in C\}\,,$$

the collection of ordered triples. Similarly, one could define ordered quadruples, etc. The n-*fold* Cartesian product of a set A with itself is

$$A^n := \underbrace{A \times \cdots \times A}_{n \text{ times}}\,.$$

For example, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is ordinary 3-space.