Math 112 lecture for Friday, Week 2

Equivalence classes

(Supplemental reading: Section 2.3 in Swanson.)

Definition. Let $\sim$ be an equivalence relation on a set S. The *equivalence class* for $x \in S$ is
$$[x] := \{y \in S : y \sim x\}.$$
The *quotient* of S by $\sim$ is the set of equivalence classes for $\sim$:
$$S/\!\sim\, := \{[x] : x \in S\}.$$
We also refer to $S/\!\sim$ as "S modulo $\sim$" or "S mod $\sim$".

Last time, we introduced an equivalence relation on $\mathbb{Z}$ for each $n \in \mathbb{Z}$. Fix your favorite integer n. Then for $a, b \in \mathbb{Z}$ we will say $a \sim b$ if a and b differ by a multiple of n, i.e., if
$$a - b = kn$$
for some $k \in \mathbb{Z}$. We use the notation $\mathbb{Z}/n\mathbb{Z}$ to denote $\mathbb{Z}/\!\sim$, the set of equivalence classes of $\mathbb{Z}$ modulo n.

Example. Consider the equivalence relation $\mathbb{Z}$ we defined above for the case $n = 2$. There are two equivalence classes:
$$\begin{aligned}[0] &= \{0, \pm 2, \pm 4, \dots\}\\ [1] &= \{1, \pm 3, \pm 5, \dots\}.\end{aligned}$$
The "name" of an equivalence class is not necessarily unique. In this example, we have $[0] = [2]$ and $[1] = [-17]$, for instance. Note that people use these equivalence classes all the time: it's just the notion of even and odd.

Example. What are the equivalence classes when $n = 3$? Looking above, we see that there are three of them:
$$\begin{aligned}[0] &= \{\dots, -6, -3, 0, 3, 6, \dots\}\\ [1] &= \{\dots, -5, -2, 1, 4, 7, \dots\}\\ [2] &= \{\dots, -4, -1, 2, 5, 8, \dots\}.\end{aligned}$$
It's interesting that, unlike the case $n = 2$, there aren't common words for the three equivalence classes of the integers modulo three.

Proposition. Let $n \in \mathbb{Z}$ and for $a, b \in \mathbb{Z}$, say $a \sim b$ if

$$a - b = kn$$

for some integer k. Then $\sim$ is an equivalence relation.

Proof. We need to show that $\sim$ is reflexive, symmetric, and transitive. Let $a, b, c \in \mathbb{Z}$.

Reflexivity. We have $a \sim a$ since $a - a = 0 \cdot n$. (We are letting $k = 0$ in the definition of $\sim$.)

Symmetry. Suppose that $a \sim b$. This means that there exists a $k \in \mathbb{Z}$ such that

$$a - b = kn.$$

But then

$$b - a = (-k)n.$$

Hence $b \sim a$.

Transitivity. Suppose that $a \sim b$ and $b \sim c$. Then there exist $k, \ell \in \mathbb{Z}$ such that

$$a - b = kn \quad \text{and} \quad b - c = \ell n.$$

Adding these two equations, we get

$$a - c = (a - b) + (b - c) = kn + \ell n = (k + \ell)n.$$

Therefore, $a \sim c$. To help make this last point as clear as possible, we can let $m := k + \ell$. Then $m \in \mathbb{Z}$, and

$$a - c = mn.$$

$\square$

Template. Here is a template for a proof that a given relation is an equivalence relation:

Proposition. Define a relation $\sim$ on a set A by blah, blah, blah. Then $\sim$ is an equivalence relation.

Proof. Let $a, b, c \in A$.

Reflexivity. We have $a \sim a$ since blah, blah, blah. Therefore, $\sim$ is reflexive.

Symmetry. Suppose that $a \sim b$. Then, blah, blah, blah. It follows that $b \sim a$. Therefore $\sim$ is symmetric.

Transitivity. Suppose that $a \sim b$ and $b \sim c$. Then blah, blah, blah. It follows that $a \sim c$. Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation. $\square$