Math 112 Lecture for Wednesday, Week 1

(Supplemental reading: Sections 1.4 and 1.5 in Swanson.)

Our first goal is to learn how to write a perfect proof by induction. (This material overlaps with that in Math 113, but it's important enough to go over twice.)

**Example (template).**

**Proposition 1.** For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

We'll discuss this theorem a bit before proving it by induction. For the case $n = 3$, it says:

$$1 + 2 + 3 = \frac{3 \cdot 4}{2},$$

and it's easy to see that both sides of this equation are equal to 6. When $n = 4$, we have

$$1 + 2 + 3 + 4 = 10 = \frac{4 \cdot 5}{2}.$$

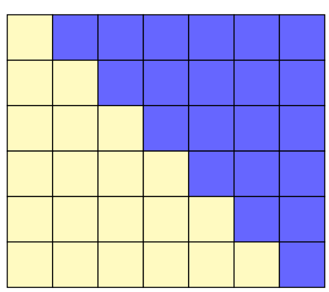
There is a tricky way to prove this theorem without using induction. Consider the following for the case $n = 6$:

$$\begin{array}{rc} & 1+2+3+4+5+6 \\ + & 6+5+4+3+2+1 \\ \hline & 7+7+7+7+7+7 \end{array} = 6 \cdot 7$$

Adding the sum twice gives $6 \cdot 7 = 42$. Divide by two to get the sum:

$$1 + 2 + 3 + 4 + 5 + 6 = \frac{6 \cdot 7}{2} = 21.$$

In a picture:



The $6 \times 7$ square contains our sum twice—once in yellow and once in blue. The proof clearly generalizes:

$$
\begin{array}{rccccccccc}
 & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\
+ & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\
\hline
 & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1)
\end{array}
\quad = n \cdot (n+1).
$$

Divide by two to get the general sum formula:

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

We now give a proof of the proposition *using induction*. Please use it as a template for your own induction proofs.

*Proof.* We will prove this by induction. For the base case, $n = 1$, the result holds since in that case

$$1 + 2 + \cdots + n = 1 = \frac{1 \cdot (1+1)}{2}.$$

Suppose the result holds for some $n \geq 1$. It follows that

$$
\begin{aligned}
1 + 2 + \cdots + (n+1) &= (1 + 2 + \cdots + n) + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by the induction hypothesis)} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2))}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}.
\end{aligned}
$$

So the result holds for the case $n+1$, too. The proposition follows by induction. $\square$

**Rules.**

1. Always start a proof by induction by telling your reader that you are giving a proof by induction.

2. Next, show that result holds for the smallest value of $n$ in question—in this case, $n = 1$.

3. Note that we assume the result is true *for some* $n \geq 1$. If we said, instead: "assume the result holds for $n \geq 1$", this would mean we're assuming the result for all $n \geq 1$. But that would be circular: we'd be assuming what we are trying to prove. Instead, at the induction step, we are merely saying that *if* we did know the result for a particular value of $n$, we could prove that it follows for the next value of $n$.

4. End the proof with a $\square$. This tells the reader the proof is over.

**Summation notation.** For integers $m \leq n$, and a function $f$ defined at the integers $m, m+1, \dots, n$, we use the notation

$$\sum_{k=m}^{n} f(k) := f(m) + f(m+1) + \cdots + f(n).$$

(Note: the notation $A := B$ means $A$ is *defined* to be $B$.) A closer look:

$$\sum_{k=m}^{n}$$

**upper bound: sum stops here**

**Greek letter sigma for "sum"**

**dummy variable**

**lower bound: sum starts here**

**Example.** Suppose that $f(k) = k^2$. Then

$$\begin{aligned}\sum_{k=-1}^{2} f(k) &= f(-1) + f(0) + f(1) + f(2)\\ &= (-1)^2 + 0^2 + 1^2 + 2^2\\ &= 1 + 0 + 1 + 4\\ &= 6.\end{aligned}$$

For this same sum we could write $\sum_{k=-1}^{2} k^2$ or $\sum_{t=-1}^{2} t^2$, for example.

**Note:** If $m > n$, then by convention, we take $\sum_{k=m}^{n} f(k) := 0$. This is called the *empty sum*.

**More examples.**

$$\sum_{i=2}^{4}(2i+i^2) = (2\cdot 2+2^2)+(2\cdot 3+3^2)+(2\cdot 4+4^2) = 47$$

$$\sum_{k=1}^{5} 2 = 2+2+2+2+2 = 10.$$

In general,

$$\sum_{k=m}^{n}(af(k)+bg(k)) = a\sum_{k=m}^{n} f(k) + b\sum_{k=m}^{n} g(k).$$

**Product notation.** There is a similar notation for products:

$$\prod_{k=m}^{n} f(k) := f(m)\cdot f(m+1)\cdots f(n).$$

For example,

$$\prod_{k=1}^{n} k = 1\cdot 2\cdots n =: n!.$$

If $m > n$, we define the *empty product* by $\prod_{k=m}^{n} f(k) := 1$. (So, for example, $0! = 1$.)

**Back to induction.** We now give our induction proof of Proposition 1 using summation notation:

**Proposition 1 (using summation notation).** For $n \geq 1$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

*Proof.* We will prove this by induction. The base case holds since

$$\sum_{i=1}^{1} i = 1 = \frac{1\cdot(1+1)}{2}.$$

Suppose the result holds for some $n \geq 1$. Then

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1)$$

$$
\begin{aligned}
&= \frac{n(n+1)}{2} + (n+1) && \text{by the induction hypothesis} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}.
\end{aligned}
$$

So the result then holds for $n+1$, too. The result holds for all $n \geq 1$, by induction. $\square$

**Another example.**

**Proposition 2.** Show

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for $n \geq 1$.

*Proof.* We will prove this by induction. The base case holds since

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6}.$$

Suppose the result holds from some $n \geq 1$:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= 1^2 + \cdots + n^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 && \text{(by equation (1))} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.
\end{aligned}$$

Thus, the result then holds for $n+1$, too. Our result follows for all $n \geq$ by induction.

$\square$