Math 112 lecture for Friday, Week 1

SETS

A *set* is a collection of objects. The objects in a set are the set's *elements* or *members*. If A is a set, we write $m \in A$ if m is an element of A and $m \notin A$ if m is not in A.

Examples.

1. $A = [0, 1)$ is the set of real numbers x such that $0 \leq x < x$.

2. $\{1, 2, 3\}$ has three elements: 1, 2, and 3.

3. Note that
$$\{0, 1, 0, 0, 2, 2, 1, 1\} = \{0, 1, 2\} .$$
Sets don't contain the same element twice.

4. The elements of sets do not need to be numbers, e.g., {☺, ☹, 17}.

5. $A = \{\{1, 2, 3\}\}$ has only one element—the set $\{1, 2, 3\}$. Thus, we can write
$$\{1, 2, 3\} \in A \quad \text{and} \quad 1 \notin A.$$

The empty set. The *empty set* is the set with no elements. It is denoted by $\emptyset$, and we write $\emptyset = \{\}$. For any object x, we have $x \notin \emptyset$. For example, $\emptyset \notin \emptyset$. On the other hand, note that the set $\{\emptyset\}$ is not empty—it has one element, namely, $\emptyset \in \{\emptyset\}$.

Propositional definition of a set. Let A be a set, and let $P(x)$ be a proposition—a statement that is either true or false—that depends on elements $x \in A$. We use the following notation sometimes to define sets
$$\{x \in A : P(x)\} .$$
We read this: "the set of elements x in the set A such that $P(x)$ is true. The colon, :, translates as "such that". Some examples:
$$\begin{aligned} \{x \in \mathbb{R} : 0 \leq x < 1\} &= [0, 1) \\ \left\{x \in \mathbb{R} : x^2 = -1\right\} &= \emptyset \\ \left\{x \in \mathbb{R} : x^2 = 1\right\} &= \{-1, 1\} . \end{aligned}$$

Some special sets:

$$
\begin{array}{rl}
\text{natural numbers} & \mathbb{N} := \{0, 1, 2, \dots\} \\
\text{positive natural numbers} & \mathbb{N}^+ := \mathbb{N}_{>0} := \{1, 2, \dots\} \\
\text{integers} & \mathbb{Z} := \{\dots, -2, -1, 01, 2, \dots\} \\
\text{rational numbers} & \mathbb{Q} := \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\} \\
\text{real numbers} & \mathbb{R} \quad \text{(more about these soon)} \\
\text{complex numbers} & \mathbb{C} \quad \text{(more about these soon).}
\end{array}
$$

Subsets and operations.

Definition. Let A and B be sets. Then A is a *subset* of B, denoted $A \subseteq B$, if every element of A is an element of B. If $A \subseteq B$, we may use the notation $B \supseteq A$ and say that B is a *superset* of A: so A is a subset of B if and only if B is a superset of A.

Time out for a remark about notation: In mathematics, the symbol "$\Rightarrow$" means "implies". Please use this rule: only use this symbol in your writing if it can be replaced by the word "implies". It is common for people who are just beginning to write mathematics to use the symbol to mean "equals" or "the next step in the argument is". That's not acceptable from now on.

Using this notation, we can write that $A \subseteq B$ if

$$x \in A \quad \Rightarrow \quad x \in B.$$

For some examples of subsets, we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Definition. Let A and B be sets. Then $A = B$ if (i) $A \subseteq B$, and (ii) $B \subseteq A$. On the other hand, A is a *proper subset* of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In that case, we might write $A \subset B$ (although some texts do not make the distinction in notation between $\subseteq$ and $\subset$).

Examples. We have $A \subseteq A$, and $(0, 1) := \{x \in \mathbb{R} : 0 < x < 1\} \subsetneq \mathbb{R}$. We also have $\emptyset \subseteq A$ since otherwise there would need to be an element of $\emptyset$ that in not in A. That's impossible since $\emptyset$ has no elements.

Warning: Don't confuse $\subseteq$ with $\in$. For example, $1 \in \{1, 2\}$ but it is **not** true that $1 \subseteq \{1, 2\}$. Instead, $\{1\} \subseteq \{1, 2\}$.

Definition. The *intersection* of sets A and B is

$$A \cap B := \{x : x \in A \text{ and } x \in B\},$$

the set of elements that are in both A **and** B. Their *union* is

$$A \cup B := \{x : x \in A \text{ or } x \in B\},$$

the set of elements that are in A or B (or both[1]). The *complement*[2] of B in A is

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\},$$

the set of elements in A but not in B. The sets A and B are *disjoint* if

$$A \cap B = \emptyset.$$

Proofs for statements involving sets.

Many theorems in mathematics are, abstractly, statements that one set is contained in another. There is a standard style for these proofs that goes like this:

Template.

Proposition If ...some hypotheses go here ..., then

$$A \subseteq B.$$

Proof. Let $a \in A$. Then [use hypotheses, definitions, calculations here]. Therefore, $a \in B$. □

Here is an example where we put the template to use. I've put comments in red in the proof which are just meant to explain what is going on. These comments would be omitted in an actual proof.

Proposition. Let A, B, C be sets with $A \subseteq B$. Then

$$A \cap C \subseteq B \cap C.$$

[1] "or" is always inclusive in mathematical writing

[2] Note the difference in spelling between "complement" and "compliment"—these words mean different things.

Proof. Let $x \in A \cap C$. Then $x \in A$ and $x \in C$. [Here, I've just gone back to the definition of $\cap$.] Since $A \subseteq B$ and $x \in A$, it follows that $x \in B$. [That's from the definition of $\subseteq$.] So now we know that $x \in B$ and $x \in C$. Therefore, $x \in B \cap C$. [Definition of $\cap$, again.]. □

Leaving out my comments, the actual proof would be:

Proof. Let $x \in A \cap C$. Then $x \in A$ and $x \in C$. Since $A \subseteq B$, and $x \in A$ is follows that $x \in B$. So now we know that $x \in B$ and $x \in C$. Therefore, $x \in B \cap C$. □

Here is another simple example of a proof of this form:

Proposition. Let A, B be sets. Then

$$A \cap B \subseteq A \cup B.$$

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since x is in A, it follows that x is in A or B. Hence, $x \in A \cup B$. □

There is so little to prove in the last proposition, that you might be confused about what you really need to say. The point I am trying to get across is, roughly, start with an element of the set on the left-hand side, spell out the definitions, and show that the element is necessarily in the set on the right-hand side.

Template.

Proposition If ...some hypotheses go here ..., then

$$A = B.$$

Proof. Let $a \in A$. Then [use hypothesese, definitions, calculations here]. Therefore, $a \in B$. Hence, $A \subseteq B$.

Conversely, let $b \in B$. Then [use hypothesese, definitions, calculations here]. Therefore, $b \in A$. Hence, $B \subseteq A$, too. Therefore, $A = B$. □

An example:

Proposition Let A and B be sets, and let $C := A \cup B$. Suppose $A \cap B = \emptyset$. Then

$$A = C \setminus B.$$

Proof. Let $a \in A$. Then $a \in C = A \cup B$. Since $A \cap B = \emptyset$ and $a \in A$, it follows that $a \notin B$. In sum, $a \in C$ and $a \notin B$. Therefore, $a \in C \setminus B$.

Conversely, let $x \in C \setminus B$. This means that $x \in C$ and $x \notin B$. But $x \in C = A \cup B$, means that $x \in A$ or $x \in B$. Since $x \notin B$, it follows that $x \in A$. □