Problem 1. Let
$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n6^n}(x-2)^n.$$
Describe the set of all points in $\mathbb{R}$ at which g converges (it will be an interval centered at 2). At which points does it converge absolutely, and at which does it converge conditionally? (Don't forget to check the endpoints of the interval of convergence.)

Problem 2. Let $\alpha \in \mathbb{R}$. The following notation will be useful: for each integer $k \geq 0$, let
$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}.$$
It might help to look up the binomial theorem (which will be the special case of this problem where α is an integer).

(a) Compute the Taylor series for $f(z) = (1+z)^\alpha$ centered at 0 (using the notation introduced above).
(b) What does your formula say in the case $\alpha = 4$? (In that case, the Taylor series will only have a finite number of nonzero terms.)
(c) What is the radius of convergence the Taylor series for f when $\alpha \notin \mathbb{Z}$? (When $\alpha \in \mathbb{Z}$, the Taylor series has only finitely many terms and always converges.)
(d) For each $n = 1, 2, 3, 4$, use the n-th order Taylor polynomial for f with $\alpha = 1/2$, to find fractions approximating $1/\sqrt{2}$ for $n = 1, \ldots, 4$. Use a computer to expand these fractions and $1/\sqrt{2}$ as decimals to see how well these approximations work.

Problem 3. Using our power series definitions of $\cos(z)$ and $\sin(z)$ for $z \in \mathbb{C}$, we have seen that $\cos'(z) = -\sin(z)$ and $\sin'(z) = \cos(z)$. Use these facts to prove that
$$\cos^2(z) + \sin^2(z) = 1$$
by completing the following steps. Let $f(z) := \cos^2(z) + \sin^2(z)$. Then: (i) using the chain rule, show that $f'(z) = 0$ and, thus, that f is a constant function, i.e. $f(z) = \alpha$ for some $\alpha \in \mathbb{C}$, and (ii) determine α by evaluating $f(z)$ at any convenient point.

Problem 4. (Exponentiation for complex numbers.) For this problem, fix a branch of the complex logarithm as we did in the group problems for Monday, Week 13. Thus, for $z \in \mathbb{C} \setminus \{0\}$, if $z = re^{i\theta}$ with $r \in \mathbb{R}_{>0}$ and $\theta \in (-\pi, \pi]$, then
$$\ln(z) = \ln(r) + i\theta.$$
Given $z, w \in \mathbb{C}$ with $z \neq 0$, define
$$z^w := e^{w\ln(z)}.$$
(Note that if $x, y \in \mathbb{R}$ with $x > 0$, this definition coincides with the usual one since $e^{y\ln(x)} = e^{\ln(x^y)} = x^y$.)

Compute the following (you will find they are both real numbers!):

(a) i^i
(b) $(-1)^i$.