The group problems from Wednesday and Friday of Week 11 may be useful.

Problem 1. Find $\lim_{x\to 3}(x^2+x+4)$ and provide an ε-δ proof. (Hint: at some point, you will need to factor a quadratic in order to find $|x-3|$.)

Problem 2. We have encountered two notions for limits: one for sequences (using ε-N) and one for functions (using ε-δ). This problem explains the connection. Suppose that $\lim_{x\to a} f(x) = L$, and let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} x_n = a$ and such that no x_n equals a. Show that the sequence $\{f(x_n)\}$ converges to L. In your proof be sure to point out where the fact that no x_n equals a is used. (The converse of this result also holds but you are not asked to prove that here.) Advice: on scratch paper, write the ε-δ definition for $\lim_{x\to a} f(x) = L$ and the ε-N definitions for $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) = L$.

Problem 3. Suppose that f and g are functions that are differentiable at some point a. Use the definition of the derivative, the definition of $f+g$, and our limit theorems for functions to prove that $f+g$ is differentiable at a and that $(f+g)'(a) = f'(a) + g'(a)$.

Problem 4. Consider the function

$$\begin{aligned} f\colon (0,\infty) &\to \mathbb{R} \\ x &\mapsto \frac{1}{x^2}. \end{aligned}$$

Use the definition of the derivative and our limit theorems to prove that f is differentiable at every point $x \in (0,\infty)$ and to compute $f'(x)$.

Problem 5. Compute the radius of convergence for each of the following series. (As always, show your work.)

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n} z^n \qquad \text{(b)}\ \sum_{n=0}^{\infty} \frac{n^3}{n!} z^n \qquad \text{(c)}\ \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} z^n$$

Problem 6. Consider an arbitrary polynomial in n of degree $d \geq 0$:

$$p(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0,$$

where the c_i are constants and $c_d \neq 0$. What is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} p(n) z^n$? (Feel free to use one of our ratio tests since once n is large, $p(n)$ will be nonzero.)