Math 112 Homework, Friday Week 10

PROBLEM 1. State whether each of the following is true or false for a real sequence. If true, give a justification. If false, give the simplest and most concrete counterexample you can think of. "Monotone" means either monotone increasing or monotone decreasing.

(a) If a bounded sequence is monotone, it's convergent.
(b) If a convergent sequence is monotone, it's bounded.
(c) If a convergent sequence is bounded, then it's monotone.
(d) If a sequence is bounded, then it's convergent.

PROBLEM 2. In order to evaluate the expression

$$\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}},$$

define a real sequence by $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3+a_n}$, and then consider $\lim\limits_{n\to\infty} a_n$.

(a) Use induction to prove that the sequence in monotone increasing.
(b) Use induction to prove that the sequence is bounded above.
(c) By the monotone convergence theorem, we conclude that $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{R}$. Proceed as in our notes to find the value of a.

PROBLEM 3. The point of this problem is to show that every convergent sequence is a Cauchy sequence (see the reading for Friday, Week 9). Let $\{a_n\}$ be a sequence complex numbers, and suppose that $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{C}$. Use an $\varepsilon/2$-argument to prove that $\{a_n\}$ is a Cauchy sequence. (Pointers: Start by fixing $\varepsilon > 0$. Use the definition of $\lim_{n\to\infty} a_n = a$ but with $\varepsilon/2$. Next, look at the definition of a Cauchy sequence and use the triangle inequality. The idea is that all points in the sequence get close to a eventually and hence get close to each other. Note that $a_m - a_n = (a_m - a) - (a_n - a)$.)

PROBLEM 4. Let $\{a_n\}$ be a Cauchy sequence of complex numbers. Prove that $\{a_n\}$ is bounded, directly from the definition of a Cauchy sequence. (Idea: Let $\varepsilon = 1$ and use the fact that the sequence is Cauchy to get an N such that $m, n > N$ imply $|a_m - a_n| < \varepsilon = 1$. In particular, this implies that $|a_{N+1} - a_n| < 1$ for all $n > N$. In other words, a_n is in the ball of radius 1 centered at a_{N+1} whenever $n > N$. Now try to imitate the proof that convergent sequences are bounded given in our notes but using a_{N+1} in place of the limit of the sequence. You will need the reverse triangle inequality.)

PROBLEM 5. Use the geometric series test to determine whether each of the following series converges. If a series does converge, find its limit.

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{8}\right)^n$.

(b) $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{3^{n+2}}{10^n}$. Note that the sum starts at $n = 3$.

(c) $\displaystyle\sum_{n=0}^{\infty} \pi^n e^{-n}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4} + \frac{\sqrt{3}}{4}i\right)^n$. Express your answer in the form $a + bi$.

First state whether why the sequence converges or diverges, and then find the sum, if it exists.