Note: For homework from now on (unless specified), we assume all of the basic arithmetic for $\mathbb{R}$ and $\mathbb{C}$ and the obvious properties having to do with inequalities for $\mathbb{R}$ without having to justify them using the field and order axioms.

PROBLEM 1. Let $z, w \in \mathbb{C}$. Say $z = a + bi$ and $w = c + di$ with $a, b, c, d \in \mathbb{R}$. Prove that $\overline{zw} = \overline{z}\,\overline{w}$.

PROBLEM 2. Compute the following and express your answers in the form $a+bi$ with $a, b \in \mathbb{R}$:

(a) $\overline{4 - 8i}$.
(b) $|3 - 4i|$.
(c) $(1 - 2i)^2$.
(d) $\mathrm{Im}(2 + 5i + i(3 - 7i) + 17)$.
(e) $(4 + 3i)/(3 + 2i)$.

PROBLEM 3. Let F be an ordered field or the complex numbers. In class, we proved the triangle inequality:

$$|u + v| \leq |u| + |v| \tag{1}$$

for all $u, v \in F$. It turns out that easy substitutions for u and v yield the useful *reverse triangle inequality*:

$$|x - y| \geq \big|\,|x| - |y|\,\big|$$

for all $x, y \in F$.

We prove the reverse triangle inequality in two steps, first proving that $|x - y| \geq |x| - |y|$ and then proving that $|x - y| \geq |y| - |x|$ for all $x, y \in F$. The result then clearly follows. At no point in the following should you revert to using the definition of $|\ |$, which is, after all, defined differently for an ordered field and for $\mathbb{C}$.

(a) Find substitutions for u and v that transform the ordinary triangle inequality, (1), into the inequality $|x - y| \geq |x| - |y|$. (The substitutions will be simple expressions involving x and y. Hint: note that our objective is equivalent to $|x| \leq |x - y| + |y|$.)
(b) Use part (a) and the fact that $|-s| = |s|$ for all $s \in F$ to show $|x - y| \geq |y| - |x|$.

PROBLEM 4. Give the polar forms for the five solutions to $z^5 = 32$.

PROBLEM 5. Let $z = 8 + 9i$ to polar form. Use a calculator to compute the approximate value of $\arg(z)$ in *degrees*, and then use that to approximate the polar form for z.

PROBLEM 6. Let D be a nonempty subset of $\mathbb{R}$. Let $f\colon D \to \mathbb{R}$ and $g\colon D \to R$ be functions. Recall the notation

$$f(D) := \{f(x) : x \in D\} \quad \text{and} \quad g(D) := \{g(x) : x \in D\}\,.$$

Define $h\colon D \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. (This function h is usually denoted $f + g$, for obvious reasons). Suppose that $f(D)$ and $g(D)$ are bounded above (so their suprema exist by completeness of $\mathbb{R}$).

(a) Show that $h(D)$ is bounded above by $\sup f(D) + \sup g(D)$. (Start: Let $y \in h(D)$. Therefore, $y = h(x)$ for some $x \in D$.)

(b) Since $h(D)$ is bounded above, it has a supremum by completeness of $\mathbb{R}$. Show that $\sup h(D) \leq \sup f(D) + \sup g(D)$.

(c) Find two specific functions $f, g : [-1, 1] \to \mathbb{R}$ such that we have a strict inequality $\sup h([-1, 1]) < \sup f([-1, 1]) + \sup g([-1, 1])$.