PROBLEM 1. Recall that the *identity function* on a set A is the function $\mathrm{id}_A\colon A \to A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$. In the reading, it was mentioned that a function $f\colon A \to B$ is a bijective if and only if there is a function $g\colon B \to A$ such that both $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. (In this case, g is the inverse of f.) The point of this problem is to see that both of these conditions are necessary.

(a) Suppose $f\colon A \to B$ and $g\colon B \to A$ are functions.

  i. Show that if $g \circ f = \mathrm{id}_A$, then f is injective. (In this case, g is called a *left inverse* for f.

  ii. Show that if $f \circ g = \mathrm{id}_B$, then f is surjective. (In this case, g is called a *right inverse* for f.)

(b) Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. Define two functions $f\colon A \to B$ and $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and f is not a bijection.

PROBLEM 2. Let $f\colon A \to B$ be a function between sets A and B. Let X and Y be subsets of B. Prove that

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$$

[Notes: (i) follow the template for proving two sets are equal; (ii) recall the definition of the inverse image of a set: if $Z \subseteq B$, then $a \in f^{-1}(Z)$ means that $f(a) \in Z$.]

PROBLEM 3.

(a) Let $a \in \mathbb{Z}$ have digits $a_k a_{k-1} \dots a_1 a_0$. In other words,

$$a = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k.$$

For example, $547 = 7 + 4 \cdot 10 + 5 \cdot 10^2$. Show that, in general,

$$a = a_0 + a_1 + \cdots + a_k \bmod 9.$$

(Thus, for instance, $547 = 5 + 4 + 7 = 0 + 7 = 7 \bmod 9$.)

(b) Show how the above observation allows you to easily check that the integer 12345678 is divisible by 9.

The observation in the previous problem is the basis of the common and useful way of checking arithmetic called "casting out 9s". To check that your arithmetic is correct in adding a collection of multi-digit integers, first add all the digits, casting out 9s as you go to keep the sum small. The result is the sum of the digits modulo 9. Next add the digits of your answer, again casting out 9s. If your two results don't agree modulo 9, then you made an arithmetic mistake somewhere. For instance, consider the following calculation:

$$\begin{array}{rr} & 59284 \\ + & 27968 \\ \hline & 86252 \end{array}$$

Working modulo 9,

$$\begin{aligned} 59284 &= 5 + 9 + 2 + 8 + 4 = (5 + 4) + 9 + 2 + 8 \\ &= 0 + 0 + 2 + (-1) = 1 \end{aligned}$$

$$27968 = 2 + 7 + 9 + 6 + 8 = (2 + 7) + 9 + 6 + (-1) = 0 + 0 + 6 - 1 = 5.$$

Therefore,

$$59284 + 27968 = 1 + 5 = 6 \bmod 9.$$

On the other hand, again modulo 9,

$$86252 = 8 + 6 + 2 + 5 + 2 = -1 + 6 + (2 + 5 + 2) = -1 + 6 + 9 = -1 + 6 = 5.$$

So

$$59284 + 27968 = 6 \neq 5 = 86252 \bmod 9,$$

which shows the arithmetic is faulty.

The digits of 59284 and 27968 were processed separately above, but they could have been combined: $5 + 9 + 2 + 8 + 4 + 2 + 7 + 9 + 6 + 8 =$ etc. mod 9, looking for pairs adding to 9 to discard.

Problem 4.

(a) Apply the method of casting out 9s to show the following arithmetic is mistaken. (As you go, look for digits that sum to 9 casting these out since that don't effect the sum modulo 9.)

$$\begin{array}{rr} & 183 \\ & 247 \\ & 346 \\ & 739 \\ + & 435 \\ \hline & 1960 \end{array}$$

What is the sum of the numbers modulo 9 (above the line), and what is the (incorrect) bottom-line sum modulo 9?

(b) Explain why the casting out 9s method of error-checking is not foolproof. Give a concrete example.