Math 112 Homework for Friday, Week 4

In the following, we will write $a = b \bmod n$ if $a - b = kn$ for some $k \in \mathbb{Z}$, i.e., if $\bar{a} = \bar{b}$ in $\mathbb{Z}/n\mathbb{Z}$.

1. Let a number $n \in \mathbb{Z}$ have digits $n_k n_{k-1} \ldots n_1 n_0$. In other words,

$$n = n_0 + n_1 \cdot 10 + n_2 \cdot 10^2 + \cdots + n_k \cdot 10^k.$$

   For example, $547 = 7 + 4 \cdot 10 + 5 \cdot 10^2$. Show that

$$n = n_0 + n_1 + \cdots + n_k \bmod 9.$$

2. Show how the above observation allows you to easily check that the integer 12345678 is divisible by 9.

3. The observation is the basis of the common and useful way of checking arithmetic called "casting out 9s". To check that your arithmetic is correct in adding a collection of multi-digit integers, first add all the digits, casting out 9s as you go to keep the sum small. The result is the sum of the digits modulo 9. Next add the digits of your answer, again casting out 9s. If your two results don't agree modulo 9, then you made an arithmetic mistake somewhere. For instance, suppose I made the following calculation:

$$\begin{array}{rr} & 59284 \\ + & 27968 \\ \hline & 86252 \end{array}$$

   Working modulo 9,

$$\begin{aligned} 59284 &= 5 + 9 + 2 + 8 + 4 = (5 + 4) + 9 + 2 + 8 \\ &= 0 + 0 + 2 + (-1) = 1 \\ 27968 &= 2 + 7 + 9 + 6 + 8 = (2 + 7) + 9 + 6 + (-1) = 0 + 0 + 6 - 1 = 5. \end{aligned}$$

   Therefore,

$$59284 + 27968 = 1 + 5 = 6 \bmod 9.$$

   On the other hand, again modulo 9,

$$86252 = 8 + 6 + 2 + 5 + 2 = -1 + 6 + (2 + 5 + 2) = -1 + 6 + 9 = -1 + 6 = 5.$$

   So

$$59284 + 27968 = 6 \neq 5 = 86252 \bmod 9,$$

which shows my arithmetic is faulty.

By the way, I processed the digits of 59284 and 27968 separately above, but I could have combined them all: $5+9+2+8+4+2+7+9+6+8 =$ etc. mod 9, looking for pairs adding to 9 to discard.

Problem. In a similar way, show that the following arithmetic is mistaken. (As you go, look for digits that sum to 9. You cast these out since that don't effect the sum modulo 9.)

$$\begin{array}{rr} & 183 \\ & 247 \\ & 346 \\ & 739 \\ + & 435 \\ \hline & 1960 \end{array}$$

What is the sum of the numbers modulo 9 (above the line), and what is the (incorrect) bottom-line sum modulo 9?

4. Explain why the casting out 9s method of error checking is not foolproof. Give a concrete example.

5. Show that if the digits of $n \in \mathbb{Z}$ are $n_k n_{k-1} \ldots n_1 n_0$, then

$$n = n_0 - n_1 + n_2 - n_3 + \cdots + (-1)^k n_k \bmod 11.$$

6. Find a number between 0 and 10 that is 12345678987654321 mod 11 by using the above trick.

7. What is the last digit of $3^{3^{2018}}$? (Hints: To find the last digit of a number, work modulo 10. Compute $3, 3^2, 3^3, 3^4, \ldots$ modulo 10 until a clear pattern emerges. You might eventually need to work modulo a different number. Also: note that we are talking about $3^{3^{2018}}$, not $(3^3)^{2018}$. As simpler example: $3^{3^2} = 3^9 \neq (3^3)^2 = 3^6$).