PROBLEM 1. Let $a, b, c \in \mathbb{R}$ and

$$f(x) = x^3 + ax^2 + bx + c.$$

How does the intermediate value theorem guarantee the existence of $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$? A rigorous proof is not required. Generalize this result.

Solution. We have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. So there must exist points $u, v \in \mathbb{R}$ such that $f(u) < 0$ and $f(v) > 0$. Since f is continuous, the result now follows from the IVT. The same argument works for any odd-degree polynomial.

PROBLEM 2. Let $f(x) = x^3 - 3x + 1$. Use the intermediate value theorem to prove that the equation $f(x) = 0$ has at least three solutions in $\mathbb{R}$

Solution. We have

$$f(-2) = -1 < 0, \quad f(0) = 1 > 0, \quad f(1) = -1 < 0, \quad f(2) = 3 > 0.$$

The IVT then implies that f has zeros in each of the intervals

$$(-2, 0), \quad (0, 1), \quad \text{and} \quad (1, 2).$$

PROBLEM 3. Slice the earth with a plane to get a circle. Use the intermediate value theorem to prove there are opposite points on this circle having the same temperature. (Describe the function to which you are applying the IVT and the assumptions that you are making in order for the hypotheses of the theorem to be satisfied.)

Solution. Parametrize the circle using the angle θ, and let $T(\theta)$ be the temperature at the point on the circle corresponding to θ. Then let $f(\theta) = T(\theta) - T(\theta + \pi)$ for $\theta \in [0, \pi]$. We are looking for a θ such than $f(\theta) = 0$. If $f(0) = 0$, we are done. Otherwise, note that

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -f(0).$$

Since $f(0)$ and $f(\pi)$ have opposite signs, the IVT now applies. We are assuming that temperature varies continuously around the circle.

PROBLEM 4. Suppose you hike a loop trial in the Columbia River Gorge. You stop for lunch someplace along the way, neither at the highest nor the lowest point. Under what conditions can you guarantee that there is some other point along the trail with the same elevation? (Again, carefully describe the function to which you are applying the IVT, and check the hypotheses that must be satisfied.)

Solution. Let $f(t)$ be the difference in elevation between your position at time t and your elevation at your lunch spot. *If the trail does not intersect itself*, then one segment of loop will travel between the highest and lowest points but does not contain your lunch spot. Along that segment f is negative at the highest point and positive at the lowest. By the IVT, it must be 0 at some point in between, and at this point, you will be at the same elevation as that of your lunch spot.

PROBLEM 5. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function, and suppose that $f(a) < 0$ and $f(b) > 0$. Describe an algorithm based on the intermediate value theorem that estimates a value $c \in (a, b)$ such that $f(c) = 0$. How quickly does this algorithm converge on a solution?

Solution. Use the *divide-and-conquer* algorithm. Let $c := \frac{a+b}{2}$ be the midpoint of the interval. If $f(c) = 0$, we are done. If $f(c) < 0$, we start again, this time using considering the function f restricted to the interval $[c, b]$. Since $f(c) < 0$ and $f(b) > 0$, the IVT guarantees f is zero somewhere in that interval. Otherwise, if $f(c) > 0$, we instead restrict f to $[a, c]$ and start again.

After n iterations, we have decreased the size of the interval in which we are searching by a factor of $1/2^n$, narrowing in on the point at which f vanishes. So the algorithm converges exponentially quickly.