In the following problems, we will be trying to understand the geometry of the complex exponential function

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^z. \end{aligned}$$

Recall that if $z = x + iy \in \mathbb{C}$ for some $x, y \in \mathbb{R}$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

The last expression gives the polar form for e^z: its modulus (length) is e^x and its argument (angle) is y.

Problem 1. What is the image of e^z? Is $z \mapsto e^z$ injective? surjective?

Problem 2. Describe the image of each vertical line under e^z. It may help to note that a vertical line has the form $\{x + it : t \in \mathbb{R}\}$ for some fixed $x \in \mathbb{R}$. What happens to the images of these lines as $x \to -\infty$? What happens as $x \to \infty$?

Problem 3. Describe the image of each horizontal line under e^z.

Problem 4. (Complex logarithms)

(a) Over the real numbers, we can define the natural logarithm to be the inverse of the exponential function. Why can't we do that with the complex exponential function?

(b) Fix the following horizontal strip of width 2π in the complex plane: $H := \{x + iy \in \mathbb{C} : x \in \mathbb{R} \text{ and } y \in (-\pi, \pi]\}$. Draw two copies of $\mathbb{C}$. In the first one, draw H, and in the second, draw the image K of H under e^z. By thinking about the images of horizontal lines and vertical line segments in H, picture how H is mapped to K by e^z.

(c) Why does $e^z : H \to K$ have an inverse?

(d) We call this inverse a *branch* of the logarithm and denote it by ln (keeping in mind that this definition depended on fixing a region in the plane on which e^z is injective). Suppose $w \in K$ has polar form $w = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$. In terms of r and θ, find a formula for $z \in H$ such that $e^z = w$, i.e., find a formula for $\ln(w)$.

(e) For this branch of the logarithm, compute the following: (i) $\ln(1 + i)$, (ii) $\ln(-1)$, and (iii) $\ln(x)$ for $x \in \mathbb{R}$.