In the following problems, we will be trying to understand the geometry of the complex exponential function

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^z. \end{aligned}$$

Recall that if $z = x + iy \in \mathbb{C}$ for some $x, y \in \mathbb{R}$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

The last expression gives the polar form for e^z: its modulus (length) is e^x and its argument (angle) is y.

Problem 1. What is the image of e^z? Is $z \mapsto e^z$ injective? surjective?

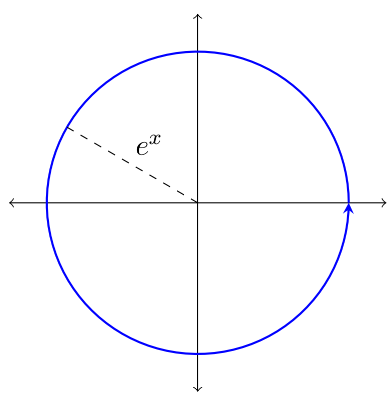
Solution. The image is $\mathbb{C} \setminus \{0\}$.

Problem 2. Describe the image of each vertical line under e^z. It may help to note that a vertical line has the form $\{x + it : t \in \mathbb{R}\}$ for some fixed $x \in \mathbb{R}$. What happens to the images of these lines as $x \to -\infty$? What happens as $x \to \infty$?

Solution. We have

$$e^{x+it} = e^x(\cos(t) + i\sin(t)),$$

which traces out a circle of radius e^x centered at the origin as t varies.



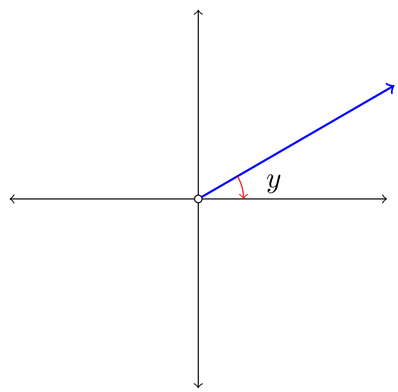
As $x \to -\infty$, the circle's radius approaches 0, and as $x \to \infty$, the circle's radius approached ∞.

Problem 3. Describe the image of each horizontal line under e^z.

Solution. We have

$$e^{t+iy} = e^t(\cos(y) + i\sin(y)),$$

which traces out a ray emanating from the origin with angle y:



PROBLEM 4. (Complex logarithms)

(a) Over the real numbers, we can define the natural logarithm to be the inverse of the exponential function. Why can't we do that with the complex exponential function?

(b) Fix the following horizontal strip of width 2π in the complex plane: $H := \{x + iy \in \mathbb{C} : x \in \mathbb{R} \text{ and } y \in (-\pi, \pi]\}$. Draw two copies of $\mathbb{C}$. In the first one, draw H, and in the second, draw the image K of H under e^z. By thinking about the images of horizontal lines and vertical line segments in H, picture how H is mapped to K by e^z.

(c) Why does $e^z : H \to K$ have an inverse?

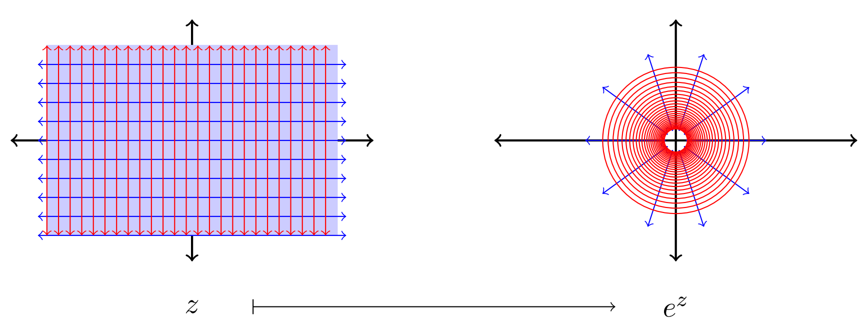
(d) We call this inverse a *branch* of the logarithm and denote it by ln (keeping in mind that this definition depended on fixing a region in the plane on which e^z is injective). Suppose $w \in K$ has polar form $w = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$. In terms of r and θ, find a formula for $z \in H$ such that $e^z = w$, i.e., find a formula for $\ln(w)$.

(e) For this branch of the logarithm, compute the following: (i) $\ln(1 + i)$, (ii) $\ln(-1)$, and (iii) $\ln(x)$ for $x \in \mathbb{R}$.

Solution.

(a) The function $z \mapsto e^z$ is not injective.

(b) The image is $K = \mathbb{C} \setminus \{0\}$.



(c) The function e^z is injective when restricted to H and then surjective by definition of K.

(d) We have

$$\begin{aligned} \ln\colon \mathbb{C} \setminus \{0\} &\to H \\ re^{i\theta} &\mapsto \ln(r) + i\theta \end{aligned}$$

where $\ln(r)$ is the ordinary natural logarithm for real numbers since $e^{\ln(r)+i\theta} = e^{\ln(r)}e^{i\theta} = re^{i\theta}$. Note that $\theta \in (-\pi, \pi]$.

(e) We have

$$\ln(1+i) = \ln\left(\sqrt{2}e^{i\frac{\pi}{4}}\right) = \ln(\sqrt{2}) + i\frac{\pi}{4}$$

$$\ln(-1) = \ln\left(1 \cdot e^{i\pi}\right) = \ln(1) + i\pi = i\pi$$

$$\ln(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(|x|) + i\pi & \text{if } x < 0. \end{cases}$$

where the logs on the right-hand side are the ordinary natural logs for real numbers.