In the lecture notes, we argued that

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots \tag{1}$$

and, by equating the coefficients of z^2 on each side, showed that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

where $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function. The point of the problems below is to show that $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Problem 1. Imagine expanding the product on the right-hand side of (1). Each term in the expansion corresponds to making a choice in each factor between either 1 or $-\frac{z^2}{m^2\pi^2}$. Give a couple of examples of terms in the expansion that contribute to the coefficient of z^4. What does the general term contributing to the coefficient of z^4 look like?

Problem 2.

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots \tag{1}$$

Evaluate the coefficient of z^4 in the expansion of the right-hand side of (1) by looking at the left-hand side. (This should be easy.)

Problem 3.

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Consider the following table

	$\frac{1}{1^2}$	$\frac{1}{2^2}$	$\frac{1}{3^2}$	$\frac{1}{4^2}$	$\cdots$
$\frac{1}{1^2}$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{3}{1^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{2^2}$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{3^2}$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{4^2}$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(a) Why is the sum of all of the entries in the table is $\zeta(2)^2$.

(b) What is the sum of the terms on the diagonal in terms of the zeta function?

(c) What is the sum of the terms off of the diagonal? (Hint: see Problems 1 and 2 in order to find a numerical value.)

PROBLEM 4. Use Problem 3 to show that
$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$