In the lecture notes, we argued that

$$1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots=\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\left(1-\frac{z^2}{9\pi^2}\right)\cdots \tag{1}$$

and, by equating the coefficients of z^2 on each side, showed that

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

where $\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function. The point of the problems below is to show that $\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

Problem 1. Imagine expanding the product on the right-hand side of (1). Each term in the expansion corresponds to making a choice in each factor between either 1 or $-\frac{z^2}{m^2\pi^2}$. Give a couple of examples of terms in the expansion that contribute to the coefficient of z^4. What does the general term contributing to the coefficient of z^4 look like?

Solution. A typical term contributing to z^4 has the form

$$\left(-\frac{z^2}{i^2\pi^2}\right)\left(-\frac{z^2}{j^2\pi^2}\right)=\frac{z^4}{i^2j^2\pi^4}$$

where $i\neq j$.

Problem 2.

$$1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots=\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\left(1-\frac{z^2}{9\pi^2}\right)\cdots \tag{1}$$

Evaluate the coefficient of z^4 in the expansion of the right-hand side of (1) by looking at the left-hand side. (This should be easy.)

Solution. We get

$$\frac{1}{5!}=\frac{1}{120}.$$

Problem 3.

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

Consider the following table

	$\frac{1}{1^2}$	$\frac{1}{2^2}$	$\frac{1}{3^2}$	$\frac{1}{4^2}$	$\cdots$
$\frac{1}{1^2}$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{3}{1^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{2^2}$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{3^2}$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{4^2}$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(a) Why is the sum of all of the entries in the table is $\zeta(2)^2$.

(b) What is the sum of the terms on the diagonal in terms of the zeta function?

(c) What is the sum of the terms off of the diagonal? (Hint: see Problems 1 and 2 in order to find a numerical value.)

Solution.

(a) We have

$$\zeta(2)^2 = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

The terms in the expansion of the product have the form

$$\frac{1}{i^2} \cdot \frac{1}{j^2},$$

with $i, j \in \mathbb{Z}_{\geq 1}$. These are exactly the entries in the table.

(b) The sum of the diagonal terms is

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(c) By Problems 1 and 2

$$\sum_{\substack{i,j \in \mathbb{Z}_{\geq 1} \\ i \neq j}} \frac{1}{i^2 j^2 \pi^2} = \frac{1}{120}.$$

The sum of the off-diagonal terms contains that sum twice. Hence, the sum of the off-diagonal terms is

$$2 \sum_{\substack{i,j \in \mathbb{Z}_{\geq 1} \\ i \neq j}} \frac{1}{i^2 j^2} = 2 \cdot \frac{\pi^4}{120}. = \frac{\pi^4}{60}$$

Problem 4. Use Problem 3 to show that

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution. By Problem 3 the sum of all entries in the table is $\zeta(2)^2$. On the other hand, we can break the sum into two parts: the off-diagonal entries, whose sum is $\pi^2/120$, and the diagonal entries, whose sum is $\zeta(4)$. Therefore,

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(2)^2 - \frac{\pi^4}{120} = \left(\frac{\pi^2}{6}\right)^2 - \frac{\pi^4}{60} = \left(\frac{1}{36} - \frac{1}{60}\right)\pi^4 = \left(\frac{10-6}{360}\right)\pi^4 = \frac{\pi^4}{90}.$$