PROBLEM 1. Consider the geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

(a) Compute $zf'(z)$ in two ways and use the result to evaluate $\sum_{n=0}^{\infty} n \left(\frac{2}{3}\right)^n$.

(b) Let $g(z) = zf'(z)$. Thinking of $g(z)$ as both a power series and as a rational function, compute $zg'(z)$ in two ways. Use the result to evaluate $\sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n$.

PROBLEM 2. Define complex power series by

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$

$$C(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$S(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots.$$

Each has radius of convergence $R = \infty$ (which is easy to check with the ratio test.). Prove the following:

$$E'(z) = E(z), \quad C'(z) = -S(z), \quad \text{and} \quad S'(z) = C(z).$$

PROBLEM 3. Using the definitions from the previous problem, prove that

$$E(iz) = C(z) + iS(z).$$

Since the series involved are absolutely convergent on $\mathbb{C}$, as far as algebra goes, you can treat them like polynomials, freely rearranging their terms.

PROBLEM 4. If there is extra time, try proving that $E(w+z) = E(w)E(z)$. You might first check that the constant terms are the same, then that the order 1 terms are the same, then the order 2 terms, etc. How far can you get? Or you could try proving it all at once. The binomial theorem may be of help:

$$(w+z)^n = \sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} w^{n-k} z^k.$$

The *binomial coefficients*

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n}$$

form the n-th row of Pascal's triangle. Also, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are power series with radius of convergence R, then for $|z| < R$,

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{n-k} b_k \right) z^n,$$

which results from just multiplying out $f(z)g(z)$ as if f and g were polynomials.