PROBLEM 1. Consider the geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

(a) Compute $zf'(z)$ in two ways and use the result to evaluate $\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$.

(b) Let $g(z) = zf'(z)$. Thinking of $g(z)$ as both a power series and as a rational function, compute $zg'(z)$ in two ways. Use the result to evaluate $\sum_{n=0}^{\infty} n^2\left(\frac{2}{3}\right)^n$.

Solution.

(a) Taking derivatives, we find

$$f'(z) = \sum_{n=0}^{\infty} nz^{n-1} = \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}$$

Therefore,

$$zf'(z) = \sum_{n=0}^{\infty} nz^n = \frac{z}{(1-z)^2},$$

and

$$\sum_{n=0}^{\infty} n^2\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right) f'\left(\frac{2}{3}\right) = \frac{\left(\frac{2}{3}\right)}{\left(1-\frac{2}{3}\right)^2} = 9 \cdot \frac{2}{3} = 6.$$

(b) Using our previous results, we have

$$g'(z) = \left(\sum_{n=0}^{\infty} nz^n\right)' = \sum_{n=0}^{\infty} n^2 z^{n-1}$$

and

$$\begin{aligned} g'(z) &= \left(\frac{z}{(1-z)^2}\right)' \\ &= \frac{(z)'(1-z)^2 - z\left((1-z)^2\right)'}{(1-z)^4} && \text{(quotient rule)} \\ &= \frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} \\ &= \frac{(1-z)+2z}{(1-z)^3} \\ &= \frac{1+z}{(1-z)^3}. \end{aligned}$$

Therefore,

$$zg'(z) = \sum_{n=0}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3},$$

and, evaluating at $d = 2/3$,

$$\sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n = \frac{\left(\frac{2}{3}\right)\left(1+\frac{2}{3}\right)}{\left(1-\frac{2}{3}\right)^3} = 3^3 \cdot \frac{2}{3} \cdot \frac{5}{3} = 30.$$

Problem 2. Define complex power series by

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$

$$C(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$S(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots .$$

Each has radius of convergence $R = \infty$ (which is easy to check with the ratio test.). Prove the following:

$$E'(z) = E(z), \quad C'(z) = -S(z), \quad \text{and} \quad S'(z) = C(z).$$

Solution. We have

$$E'(z) = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = E(z)$$

$$C'(z) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right)' = \sum_{n=0}^{\infty} (-1)^n \frac{2nz^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{2nz^{2n-1}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!} = -\sum_{n=0}^{\infty} (-1)^n \frac{z^{2(n+1)-1}}{(2(n-1)-1)!} = -\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$= -S(z)$$

$$S'(z) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\right)' = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = C(z).$$

Problem 3. Using the definitions from the previous problem, prove that

$$E(iz) = C(z) + iS(z).$$

Since the series involved are absolutely convergent on $\mathbb{C}$, as far as algebra goes, you can treat them like polynomials, freely rearranging their terms.

Proof.

$$\begin{aligned}
E(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)1!} \\
&= \sum_{n=0}^{\infty} (i)^{2n} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (i)^{2n+1} \frac{z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i(i)^{2n} \frac{z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n+1}}{(2n+1)!} \\
&= C(z) + iS(z).
\end{aligned}$$

□

Problem 4. If there is extra time, try proving that $E(w+z) = E(w)E(z)$. You might first check that the constant terms are the same, then that the order 1 terms are the same, then the order 2 terms, etc. How far can you get? Or you could try proving it all at once. The binomial theorem may be of help:

$$(w+z)^n = \sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} w^{n-k} z^k.$$

The *binomial coefficients*

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n}$$

form the n-th row of Pascal's triangle. Also, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are power series with radius of convergence R, then for $|z| < R$,

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{n-k} b_k \right) z^n,$$

which results from just multiplying out $f(z)g(z)$ as if f and g were polynomials.

Proof. Calculate:

$$\begin{aligned}
E(w+z) &= \sum_{n=0}^{\infty} \frac{(w+z)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{w^{n-k} z^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{w^{n-k} z^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} w^{n-k} z^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{w^{n-k}}{(n-k)!} \frac{z^k}{k!} \\
&= \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \\
&= E(w)E(z).
\end{aligned}$$

□