PROBLEM 1. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$.

Solution. Using the power series ratio test, we find

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{(2n)!}{(n!)^2}\right) \Big/ \left(\frac{(2(n+1))!}{((n+1)!)^2}\right) &= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \frac{1}{4}.\end{aligned}$$

Therefore, $R = \frac{1}{4}$.

PROBLEM 2. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^{2n}$.

Solution. Let $w := z^2$, and consider the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} w^n$. From Problem 1, we know that this latter series converges for $|w| < \frac{1}{4}$ and diverges for $|w| > \frac{1}{4}$. Since $|w| = |z|^2$, this means the radius of convergence for the original series is $R = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

PROBLEM 3. Compute the radius of convergence of $\sum_{n=0}^{\infty} n! z^n$ and of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Solution. Using the power series ratio test, we find

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = 0$$

and

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty.$$

Thus, the radius of convergence of the former series is 0 and of the latter is ∞.

PROBLEM 4. Describe the region in the complex plane where the series $\sum_{n=1}^{\infty} \frac{(5z-2)^n}{n^2 4^n}$ converges. (Don't forget to check the boundary of the region.)

Solution. By the power series ratio test we find

$$\lim_{n\to\infty} \frac{1}{n^2 4^n} \Big/ \frac{1}{(n+1)^2 4^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2 4^{n+1}}{n^2 4^n} = \lim_{n\to\infty} 4 \left(\frac{n+1}{n}\right)^2 = 4.$$

So this series converges where $|5z-2|<4$. We have

$$|5z-2|<4 \quad \Leftrightarrow \quad \left|z-\frac{2}{5}\right|<\frac{4}{5}.$$

Thus, the series converges absolutely in the open ball of radius $\frac{4}{5}$ centered at $\frac{2}{5}\in\mathbb{C}$, i.e., at the point $\left(\frac{2}{5},0\right)$.

On the boundary of the disc, where $|5z-2|=4$, the series converges absolutely:

$$\sum_{n=1}^{\infty}\left|\frac{(5z-2)^n}{n^2 4^n}\right| = \sum_{n=1}^{\infty}\frac{4^n}{n^2 4^n} = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges by the p-test.

PROBLEM 5. What is the radius of convergence of the series $f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n$. What happens on the boundary of its disc of convergence?

Solution. Apply the power series ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| \Big/ \left|\frac{(-1)^{n+1}}{n+1}\right| = 1.$$

Thus, the radius of convergence if $R=1$. What about on the boundary of the disc of radius 1 centered at the origin? We have that $f(1)$ is the alternating harmonic series, and hence converges. On the other hand, $f(-1)$ is the harmonic series, which diverges.

One may show that the series converges at every point on the boundary except for $z=-1$. (For example, use Abel's test as here with $a_n=\frac{1}{n}$ and with $-z$ in place of z.)