PROBLEM 1. Let T_n denote the n-th degree Taylor polynomial for $f(x) = x^3 - 3x^2 - x + 3$ centered at $x = 2$.

(a) Fill in the following table, and use it to compute T_n for $n = 1, 2, 3$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$f^{(n)}(2)/n!$
0			
1			
2			
3			

(b) Show that $T_3(x) = f(x)$.

(c) Use a computer to plot f, T_1, and T_2 in some interval containing 2 using a different color for each graph. Someone in each group should share their screen so that everyone can view the plot. Here is an example of plotting the same data for the function $\tan(x)$ centered at $x = 0$ using https://sagecell.sagemath.org/:

```
p = plot(tan(x),(x,-1,1),color="black")
q = plot(x+(1/3)*x^3,(x,-1,1),color="blue")
r = plot(x+(1/3)*x^3+(2/15)*x^5,(x,-1,1),color="red")
p+q+r
```

Solution.

(a) We have

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$f^{(n)}(2)/n!$
0	$x^3 - 3x^2 - x + 3$	-6	-3
1	$3x^2 - 6x - 1$	-1	-1
2	$6x - 6$	6	3
3	6	6	1

Therefore,

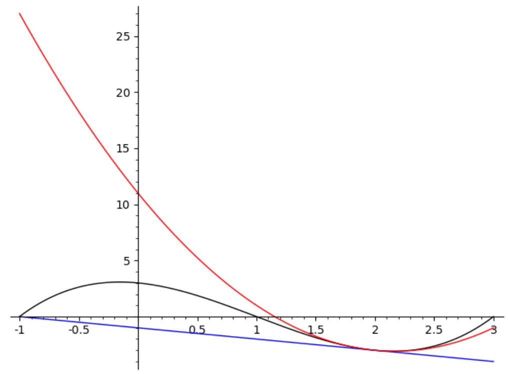
$$\begin{aligned} T_1(x) &= -3 - (x-2) \\ T_2(x) &= -3 - (x-2) + 3(x-2)^2 \\ T_3 &= -3 - (x-2) + 3(x-2)^2 + (x-2)^3. \end{aligned}$$

(b) We have

$$\begin{aligned} T_3 &= -3 - (x-2) + 3(x-2)^2 + (x-2)^3 \\ &= -3 - (x-2) + 3(x^2 - 4x + 4) + (x^3 - 6x^2 + 12x - 8) \\ &= (-3 + 2 + 12 - 8) + (-1 - 12 + 12)x + (3 - 6)x^2 + x^3 \end{aligned}$$

$$= 3 - x - 3x^2 + x^3 = f(x).$$

(c) The plots:



PROBLEM 2. Compute the first-, third-, and fifth-order Taylor polynomials for $f(x) = \sin(x)$ centered at $x = 0$ and use them to approximate $\sin(1)$. Use a computer to see how good these estimates are.

Solution. The Taylor series for $\sin(x)$ centered at 0 is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Truncating the series we find the Taylor polynomials

$$x, \quad x - \frac{x^3}{3!}, \quad \text{and} \quad 1 - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Our estimates for $\sin(1)$ are

$$1, \quad 1 - \frac{1}{6} = \frac{5}{6} = 0.8333\ldots, \quad \text{and} \quad 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} = 0.8416666\ldots$$

whereas

$$\sin(1) = 0.841470984807897\ldots$$

PROBLEM 3. Consider the function $f(x) = \dfrac{1}{x^2+1}$.

(a) Compute the Taylor series for f centered at $x = 0$. You can do this without calculating derivatives by making an appropriate substitution in the formula for the geometric series

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}.$$

(b) What is the radius of convergence for your series? Given that $f(x)$ is defined for all real numbers, can you think of a reason why its radius of convergence is not $R = \infty$?

Solution.

(a) Letting $y = -x^2$ and substituting gives

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots .$$

(b) The radius of convergence is given by the ratio test:

$$\lim_{n\to\infty} \frac{|(-1)^{n+1}x^{n+1}|}{|(-1)^n x^n|} = \lim_{n\to\infty} |x| = |x|$$

By the ratio test, the radius of convergence is 1.

(c) Thinking about f over the complex numbers, we notice that f blows up at the complex number i, which is a distance of 1 from the origin.