For convenience:

$\lim_{x\to a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

PROBLEM 1. Find $\lim\limits_{x\to 9} x^2$, and provide an ε-δ proof.

SOLUTION: Claim: $\lim_{x\to 9} x^2 = 81$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/19\}$ and suppose that $0 < |x-9| < \delta$. Then, since $\delta \le 1$, it follows that $8 < x < 10$, and hence $17 < x+9 < 19$. Combining this with the fact that $\delta \le \varepsilon/19$, we have

$$|x^2 - 81| = |x+9||x-9| < 19|x-9| < 19\delta \le 19\frac{\varepsilon}{19} = \varepsilon.$$

□

PROBLEM 2. Find $\lim\limits_{x\to 3} \dfrac{1}{2+x}$, and provide an ε-δ proof.

SOLUTION: Claim: $\lim\limits_{x\to 3} \dfrac{1}{2+x} = \dfrac{1}{5}$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, 20\varepsilon\}$ and suppose that $0 < |x-3| < \delta$. Since $\delta \le 1$, we have $2 < x < 4$, and hence $20 < 5(2+x) < 30$. Combining this with the fact that $\delta \le 20\varepsilon$, we have

$$\left|\frac{1}{2+x} - \frac{1}{5}\right| = \left|\frac{5-(2+x)}{5(2+x)}\right| = \left|\frac{3-x}{5(2+x)}\right| = \frac{|x-3|}{|5(2+x)|} < \frac{1}{20}|x-3| < \frac{1}{20}\delta \le \varepsilon.$$

□

PROBLEM 3. Find $\lim\limits_{x\to 1} \left(x^2 + 3x + 2\right)$, and provide an ε-δ proof.

SOLUTION: Claim: $\lim_{x\to 1} \left(x^2 + 3x + 2\right) = 6$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/6\}$ and suppose that $0 < |x-1| < \delta$. Then since $\delta \le 1$, we have $0 < x < 2$, and hence, $4 < x+4 < 6$. Combining this with the fact that $\delta \le \varepsilon/6$, we have

$$\begin{aligned} |x^2 + 3x + 2 - 6| &= |x^2 + 3x - 4| \\ &= |(x+4)(x-1)| \\ &= |x+4||x-1| \\ &< 6|x-1| \\ &< 6\delta \le 6 \cdot \frac{\varepsilon}{6} = \varepsilon. \end{aligned}$$

□

Problem 4. Define

$$f\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

Does $\lim_{x\to 0} f(x)$ exist? If so, then provide an ε-δ proof. If not, then provide an ε that can't be beat by any δ.

Solution: For sake of contradiction, suppose that $\lim_{x\to 0} f(x) = L$ for some $L \in \mathbb{R}$, and let $\varepsilon = 1$. Then we can find $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x) - L| < \varepsilon = 1$. There exist both a rational number $p \neq 0$ and an irrational number $q \neq 0$ within a distance of δ from 0. (For instance, we could let $p = 1/2^n$ and $q = \sqrt{2}/2^n$ for a suitably large n.) Then

$$2 = |f(p) - f(q)| = |(f(p) - L) - (f(q) - L)| \leq |f(p) - L| + |f(q) - L| < 1 + 1 = 2,$$

a contradiction.