Problem 1. Apply the ratio test to each of the following series, and state what conclusion may be drawn:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n)!}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

For part (d), you may use the fact that $\lim_{n\to\infty}(1+1/n)^n = e$.

Solution:

(a) We have

$$\frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} = \frac{(n+1)!}{n!} \cdot \frac{5^n}{5^{n+1}} = \frac{n+1}{5} \longrightarrow \infty.$$

Hence, the series diverges by the ratio test.

(b) We have

$$\frac{\frac{(n+1)^2}{(2(n+1))!}}{\frac{n^2}{(2n)!}} = \frac{(n+1)^2}{n^2} \cdot \frac{(2n)!}{(2n+2)!} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{(2n+2)(2n+1)} \longrightarrow 0.$$

Hence, the series converges by the ratio test.

(c) We have

$$\frac{\frac{1}{2(n+1)^2}}{\frac{1}{2n^2}} = \left(\frac{n+1}{n}\right)^2 \longrightarrow 1.$$

So the ratio test is inconclusive.

(d) We have

$$\begin{aligned}
\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} &= \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)!}{n!} \\
&= \frac{n^n}{(n+1)^{n+1}} \cdot (n+1) \\
&= \frac{n^n}{(n+1)^n} \\
&= \frac{1}{(1+\frac{1}{n})^n} \\
&= \frac{1}{e} < 1.
\end{aligned}$$

Hence, the series converges by the ratio test.

Problem 2. Apply the integral test to each of the following series, and state what conclusion may be drawn:

$$\text{(a)}\quad \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \qquad \text{(b)}\quad \sum_{n=1}^{\infty}\frac{1}{n^{4/3}} \qquad \text{(c)}\quad \sum_{n=1}^{\infty}\frac{n^2}{e^{n^3}}$$

Solution:

(a) We have

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{n\to\infty}\int_1^n x^{-1/2}\,dx = \lim_{n\to\infty} 2x^{1/2}\big|_1^n = 2\lim_{n\to\infty}\left(\sqrt{n}-1\right) = \infty.$$

Hence, the series diverges.

(b) We have

$$\int_1^\infty \frac{1}{x^{4/3}} = \lim_{n\to\infty}\int_1^n x^{-4/3}\,dx = -3\lim_{n\to\infty} x^{-1/3}\big|_1^n = -3\lim_{n\to\infty}\left(\frac{1}{\sqrt[3]{n}}-1\right) = 3.$$

Since the integral converges, so does the series.

(c) We have

$$\int_1^\infty \frac{x^2}{e^{x^3}}\,dx = \lim_{n\to\infty}\int_1^n x^2e^{-x^3}\,dx = -\frac{1}{3}\lim_{n\to\infty} e^{-x^3}\big|_1^n = -\frac{1}{3}\lim_{n\to\infty}\left(\frac{1}{e^{n^3}}-\frac{1}{e}\right) = \frac{1}{3e}.$$

Since the integral converges, so does the series.

Problem 3. As a consequence of our limit theorems, we know that if $\sum_n a_n$ and $\sum_n b_n$ converge, then so do $\sum_n(a_n+b_n)$ and $\sum_n ca_n$ for all constants c. It turns out that it is not necessarily true that $\sum_n a_nb_n$ converges. As a special case (where $a_n = b_n$), find a series $\sum_n a_n$ such that $\sum_n a_n = 0$, and yet $\sum_n a_n^2$ diverges to ∞.

Solution: Let $\{a_n\}$ be the sequence

$$1, -1, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}, \ldots.$$

The sequence of partial sums i

$$1, 0, \sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{3}}, 0, \ldots,$$

which converges to 0.

On the other hand, the sequence $\{a_n^2\}$ is the sequence

$$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots,$$

which diverges by comparison with the harmonic series.