Problem 1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 3z^2 + 2$. Compute $f'(3i)$ directly from the definition of the derivative.

Solution: We have

$$\begin{aligned}
\lim_{x\to a} f'(3i) &= \lim_{h\to 0} \frac{f(3i+h) - f(3i)}{h} \\
&= \lim_{h\to 0} \frac{(3(3i+h)^2 + 2) - (3(3i)^2 + 2)}{h} \\
&= \lim_{h\to 0} \frac{(3(-9 + 6ih + h^2) + 2) - (3(-9) + 2)}{h} \\
&= \lim_{h\to 0} \frac{18ih + 3h^2}{h} \\
&= \lim_{h\to 0} (18i + 3h) \\
&= 18i.
\end{aligned}$$

Problem 2. Let $A, B, C \subseteq F$ where $F = \mathbb{R}$ or $\mathbb{C}$, and suppose that $f\colon A \to B$ and $g\colon B \to C$ are continuous functions. Show that $g \circ f$ is continuous by filling in the blanks below.

Proof. Let $a \in A$, and let $\varepsilon > 0$. Since g is continuous at $f(a)$, there exists $\delta > 0$ such that

(1) $$|x - f(a)| < \delta \quad \Rightarrow \quad \boxed{}.$$

Fix this δ. Since f is continuous at a, there exists $\eta > 0$ such that

(2) $$|x - a| < \eta \quad \Rightarrow \quad \boxed{}$$

Combining (1) and (2), we see that

$$|x - a| < \eta \quad \Rightarrow \quad \boxed{}$$

Thus, $g \circ f$ is continuous at a. □

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(a)$, there exists $\delta > 0$ such that

(3) $$|x - f(a)| < \delta \quad \Rightarrow \quad |g(x) - g(f(a))| < \varepsilon.$$

Fix this δ. Since f is continuous at a, there exists $\eta > 0$ such that

(4) $$|x - a| < \eta \quad \Rightarrow \quad |f(x) - f(a)| < \delta.$$

Combining (3) and (4), we see that

$$|x-a|<\eta \quad \Rightarrow \quad |f(x)-f(a)|<\delta \quad \Rightarrow \quad |g(f(x))-g(f(a))|<\varepsilon.$$

Thus, $g \circ f$ is continuous at a. $\square$

PROBLEM 3.

(a) Let $z, w \in \mathbb{C}$. What do the triangle inequality and the reverse triangle inequality say about $|z+w|$? What about $|z-w|$?

(b) Prove that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(x) = |x|$ is continuous.

SOLUTION:

(a) Let $z, w \in \mathbb{C}$. The triangle inequality says that

$$|z+w| \leq |z| + |w|.$$

Replacing w by $-w$ in the above inequality yields

$$|z-w| = |z+(-w)| \leq |z| + |-w| = |z| + |w|.$$

Hence,

$$|z-w| \leq |z| + |w|.$$

The reverse triangle inequality says that

$$|z+w| \geq ||z| + |w||,$$

It follows that

$$|z-w| = |z+(-w)| \geq ||z| + |-w|| = ||z| + |w||,$$

i.e.,

$$|z-w| \geq ||z| + |w||.$$

(b) *Proof.* Let $a \in \mathbb{C}$. Given $\varepsilon > 0$, let $\delta = \varepsilon$, and suppose that $|x-a| < \delta$. Then, by the reverse triangle inequality,

$$|f(x)-f(a)| = ||x| - |a|| \leq |x-a| < \delta = \varepsilon.$$

$\square$