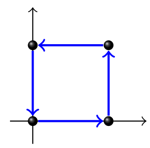
PROBLEM 1. In what sense is $\sum_{n=0}^{\infty} i^n$ a sequence? Draw this sequence in the complex plane.

Solution. The series $\sum_{n=0}^{\infty} i^n$ is the sequence of partial sums:

$$s_0 = 1,\ s_1 = 1 + i,\ s_2 = 1 + i + i^2 = 1 + i - 1 = i,\ s_3 = s_2 + i^3 = 0,$$

and $s_n = s_{n-4}$ for $n \geq 4$. This sequence is depicted below:



PROBLEM 2. Let $\{a_n\}$ be a sequence of real numbers.

(a) Critique the statement that $\sum_{n=0}^{\infty} a_n$ is convergent if and only if its sequence of partial sums is bounded. Give a proof or a counterexample for both implications.

(b) Does anything change if $\{a_n\}$ is a sequence of *nonnegative* real numbers?

Solution.

(a) For the series $\sum_{n=0}^{\infty} a_n$ to be convergent means that its sequence of partial sums is convergent. A convergent sequence is bounded. Hence, if $\sum_{n=0}^{\infty} a_n$ converges, its sequence of partial sums is bounded. For the converse, consider the series $\sum_{n=0}^{\infty} (-1)^n$. Its sequence of partial sums is $1, 0, 1, 0, \dots$ and hence is bounded. However, it does not converge.

(b) If $a_n \geq 0$ for all n, then the sequence of partial sums for $\sum_{n=0}^{\infty} a_n$ is monotonically increasing. By the monotone convergence theorem, then, the series converges if and only if its sequence of partial sums is bounded above.

PROBLEM 3. Determine whether the following series converge, and in the case one does, find its sum. If the sum is complex, express the answer in the form $a + bi$ with $a, b \in \mathbb{R}$.

$$(a)\ \sum_{n=2}^{\infty} (-1)^n \frac{3^{2n+2}}{10^n} \qquad (b)\ \sum_{n=0}^{\infty} \left(\frac{2+i}{2}\right)^n \qquad (c)\ \sum_{n=0}^{\infty} \left(\frac{3+i}{5}\right)^n.$$

Solution.

(a) We have

$$\sum_{n=2}^{\infty} (-1)^n \frac{3^{2n+2}}{10^n} = \sum_{n=2}^{\infty} (-1)^n \frac{3^{2n} \cdot 3^2}{10^n} = \sum_{n=2}^{\infty} 9 \cdot \left(-\frac{9}{10}\right)^n.$$

Since $|-9/10| < 1$, the series converge, and its value is

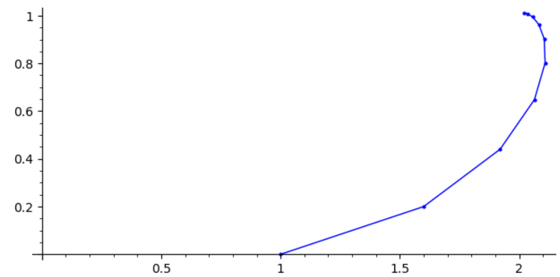
$$\sum_{n=2}^{\infty} 9 \cdot \left(-\frac{9}{10}\right)^n. = 9 \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{1}{1 - (-9/10)} = 9 \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{10}{19} = \frac{729}{190}.$$

(b) Since $|(3+i)/2| = \sqrt{10}/2 > 1$, there series diverges since its a geometric series with ratio greater than 1.

(c) Here, $|(3+i)/5| = \sqrt{10}/5 < 1$, so this geometric series is summable. The value is

$$\frac{1}{1-\frac{3+i}{5}} = \frac{5}{2-i} = \frac{5}{2-i}\cdot\frac{2+i}{(2+i)} = \frac{5(2+i)}{5} = 2+i.$$

Here is a picture of the convergence of the sequence of partial sums:



Problem 4. Express $0.99999\ldots$ as a geometric series, and sum the series. Do the same for $6.232323\ldots$ to express this number as a quotient of integers.

Solution. We have

$$\begin{aligned}
0.999\ldots &= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots \\
&= \sum_{n=1}^{\infty} 9\cdot\left(\frac{1}{10}\right)^n \\
&= 9\cdot\left(\frac{1}{10}\right)\sum_{n=0}^{\infty}\cdot\left(\frac{1}{10}\right)^n \\
&= 9\cdot\left(\frac{1}{10}\right)\frac{1}{1-1/10} \\
&= 9\cdot\left(\frac{1}{10}\right)\frac{10}{9} \\
&= 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
6.232323\ldots &= 6 + \frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \ldots \\
&= 6 + \sum_{n=1}^{\infty} 23\cdot\left(\frac{1}{100}\right)^n
\end{aligned}$$

$$= 6 + 23 \cdot \left(\frac{1}{100}\right) \sum_{n=0}^{\infty} \cdot \left(\frac{1}{100}\right)^n$$

$$= 6 + 23 \cdot \left(\frac{1}{100}\right) \frac{1}{1 - 1/100}$$

$$= 6 + 23 \cdot \left(\frac{1}{100}\right) \frac{100}{99}$$

$$= 6 + \frac{23}{99}$$

$$= \frac{617}{99}.$$

PROBLEM 5. Sum the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

Solution. Using a variant of the telescoping sum argument given in the notes, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= \frac{3}{4}.$$