PROBLEM 1. Use the limit comparison test to determine whether the following series converge. You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2 - 6n + 3}{4n^6 + n^3 + 7}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 2}}$

Solution.

(a) This series converges by limit comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ since, as $n \to \infty$,

$$\left(\frac{5n^2 - 6n + 3}{4n^6 + n^3 + 7}\right) \Big/ \left(\frac{1}{n^4}\right) = \frac{5n^6 - 6n^5 + 3n^4}{4n^6 + n^3 + 7} \longrightarrow \frac{5}{4} \neq 0.$$

(b) This series diverges by limit comparison with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ since, as $n \to \infty$,

$$\left(\frac{1}{\sqrt{n^2 + 2}}\right) \Big/ \left(\frac{1}{n}\right) = \frac{n}{\sqrt{n^2 + 2}} = \frac{\frac{1}{n} \cdot n}{\frac{1}{n} \cdot \sqrt{n^2 + 2}} = \frac{1}{\sqrt{1 + \frac{2}{n^2}}} \longrightarrow 1 \neq 0.$$

PROBLEM 2. Are the following series absolutely convergent, conditionally convergent, or divergent?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n + 1}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n + 4}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{5^{n+1}}$.

Solution.

(a) This series is divergent by the n-th term test since its sequence of terms diverges. In particular,

$$\lim_{n\to\infty} (-1)^{n+1} \frac{n}{3n + 1} \neq 0.$$

(b) This series is conditionally convergent. To apply the alternating series test, we first check that $\{\sqrt{n}/(n + 4)\}$ is (eventually) decreasing by showing the derivative with respect to n is negative. Using the quotient rule,

$$\left(\frac{n^{1/2}}{n + 4}\right)' = \frac{\frac{1}{2}n^{-1/2}(n + 4) - n^{1/2}}{(n + 4)^2} = \frac{(n + 4) - 2n}{2\sqrt{n}(n + 4)^2} = \frac{-n + 4}{2\sqrt{n}(n + 4)^2} < 0$$

for $n > 4$. Next, notice that $\lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$. (To give a formal proof of this fact, we can use the squeeze theorem since $0 \leq \frac{\sqrt{n}}{n+4} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, and we know that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.)

We have just shown that the series is convergent. It is conditionally convergent since $\sum_{n=1}^{\infty} \frac{n}{n+4}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: as $n \to \infty$,

$$\left(\frac{\sqrt{n}}{n + 4}\right) \Big/ \left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n + 4} \longrightarrow 1 \neq 0.$$

(c) This series is absolutely convergent since it is essentially a geometric series with ratio less than 1:

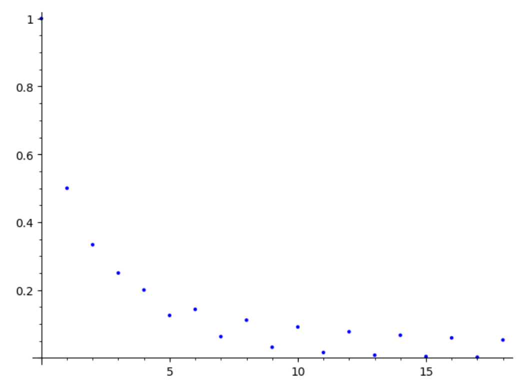
$$\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

and $3/5 < 1$.

PROBLEM 3. What does the alternating series test say about the following series?

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots$$

Solution. The alternating series test is inconclusive since the terms of the series are not monotonically decreasing. Here is a plot of some of the values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{5}, \ldots$:



PROBLEM 4. Consider the series from the previous problem:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots. \tag{1}$$

Here is a typical partial sum:

$$\begin{aligned} s_{2k+1} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \cdots - \frac{1}{2^k} + \frac{1}{2k+1} \\ &= 1 + \frac{1}{3} + \cdots + \frac{1}{2k+1} - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right). \end{aligned}$$

(a) Prove that $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ diverges to infinity.

(b) Find a lower bound for s_{2k+1} that allows you to show that the series (1) diverges.

(c) Why doesn't this example violate the alternating series test?

Solution.

(a) The series diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ since, as $k \to \infty$,

$$\left(\frac{1}{2k+1}\right) \Big/ \left(\frac{1}{k}\right) = \frac{k}{2k+1} \longrightarrow \frac{1}{2} \neq 0.$$

Since the terms of the series are positive, the partial sums for the series are monotonically increasing. Therefore, by the monotone convergence theorem, the series is not bounded. Thus, the series diverges to infinity.

(b) We have

$$\frac{1}{2} + \cdots + \frac{1}{2^k} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}\left(\frac{1}{1-1/2}\right) = 1.$$

Therefore,

$$s_{2k+1} = 1 + \frac{1}{3} + \cdots + \frac{1}{2k+1} - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right) \geq \left(1 + \frac{1}{3} + \cdots + \frac{1}{2k+1}\right) - 1$$

Since $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges to infinity, the series (1) diverges.

(c) As stated in the previous problem, the alternating series does not apply here since the term of the series are not monotonically decreasing.