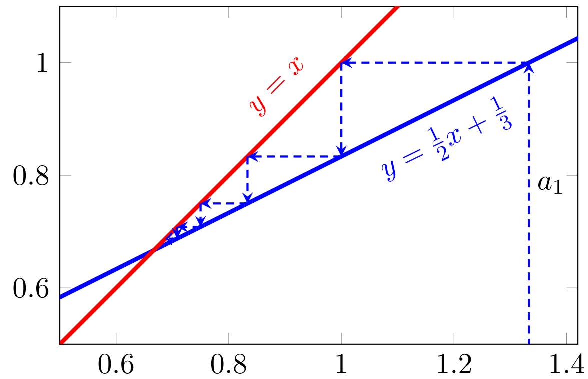
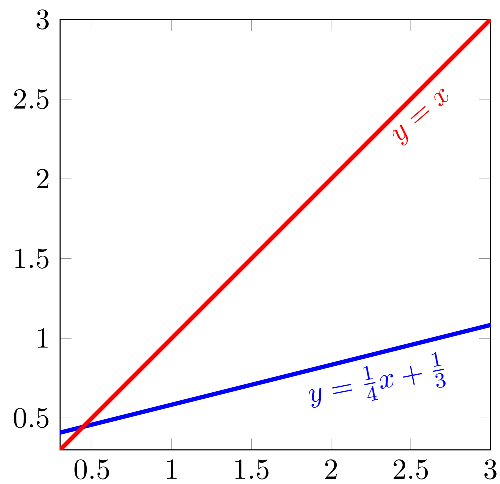
Problem 1.



(a) Following the dashed blue line, how can you see the sequence $a_1 = 4/3$ and $a_{n+1} = \frac{1}{2}a_n + \frac{1}{3}$ for $n \geq 1$ as a sequence of heights in the above diagram?

(b) Using the diagram, can you say what the behavior of the sequence would be if we started with a different value for a_1 (but used the same recursion formula for the rest of the sequence)?

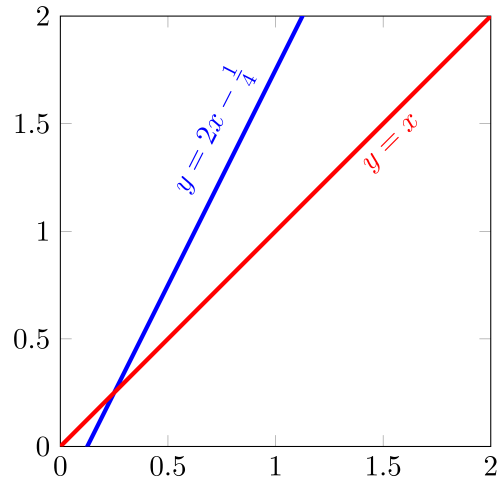
Problem 2. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = \frac{1}{4}a_n + \frac{1}{3}$.



(a) Choose a value for a_1, and draw a diagram illustrating the convergence of $\{a_n\}$.

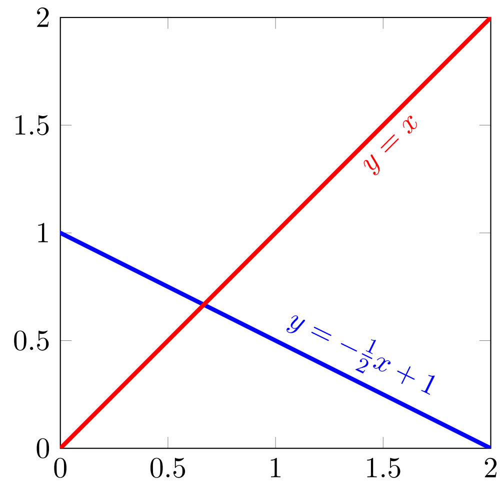
(b) Comparing this sequence with the previous one, how does the slope of the line defining the recurrence affect the rate of convergence.

PROBLEM 3. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = 2a_n - \frac{1}{4}$.



Characterize the convergence behavior of the sequence $\{a_n\}$ for each choice of initial value a_1. (Are there any special choices for a_1?)

PROBLEM 4. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = -\frac{1}{2}a_n + 1$.



Characterize the convergence behavior of the sequence $\{a_n\}$ for each choice of initial value a_1.

Problem 5. Summarize the convergence behavior of a sequence defined by $a_1 \in \mathbb{R}$ and $a_{n+1} = ma_n + b$ for $m, b \in \mathbb{R}$.

(a) Consider the cases $|m| < 1$, $|m| > 1$, $m = -1$, and $m = 1$ separately. When does the sequence converge for all initial values? When does the sequence converge for only a special initial value (in which case the sequence is constant)?

(b) When is the sequence monotone?

(c) How does $|m|$ affect the rate of convergence or divergence?

Problem 6. (Extra time.) Let $a_1 = 1$, and for $n \geq 1$, let $a_{n+1} := \frac{1}{2}a_n + \frac{1}{3}$. From the data shown below, it looks like the sequence $\{a_n\}$ is monotone decreasing and converging to $2/3$.

$$\begin{array}{llll} a_1 = 1.000000\ldots, & a_2 = 0.833333\ldots, & a_3 = 0.750000\ldots, & a_4 = 0.708333\ldots, \\ a_5 = 0.687500\ldots, & a_6 = 0.677083\ldots, & a_7 = 0.671875\ldots, & a_8 = 0.669271\ldots, \\ \quad\ldots, & a_{20} = 0.666667, & \quad\ldots & \end{array}$$

(a) Prove that $\{a_n\}$ is bounded below by $2/3$.

(b) Prove that $\{a_n\}$ is monotone decreasing.

(c) Thus, by the MCT, the sequence converges. Find its limit.