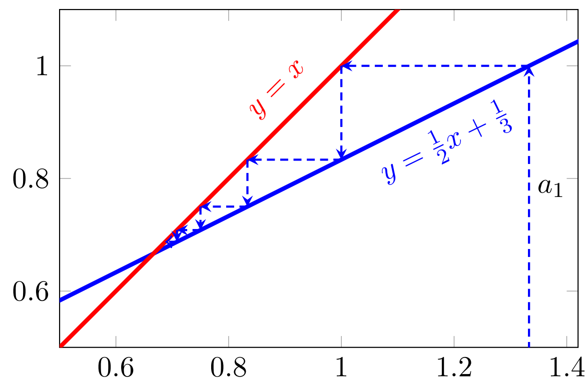
PROBLEM 1.

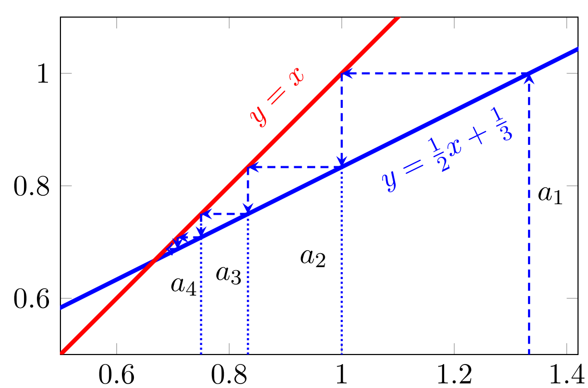


(a) Following the densely dashed blue line, how can you see the sequence $a_1 = 4/3$ and $a_{n+1} = \frac{1}{2}a_n + \frac{1}{3}$ for $n \geq 1$ as a sequence of heights in the above diagram?

(b) Using the diagram, can you say what the behavior of the sequence would be if we started with a different value for a_1 (but used the same recursion formula for the rest of the sequence)?

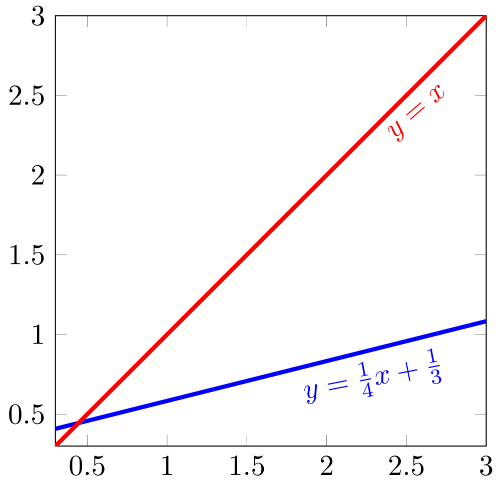
Solution.

(a)



(b) With any initial point, the sequence will converge to 2/3.

PROBLEM 2. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = \frac{1}{4}a_n + \frac{1}{3}$.

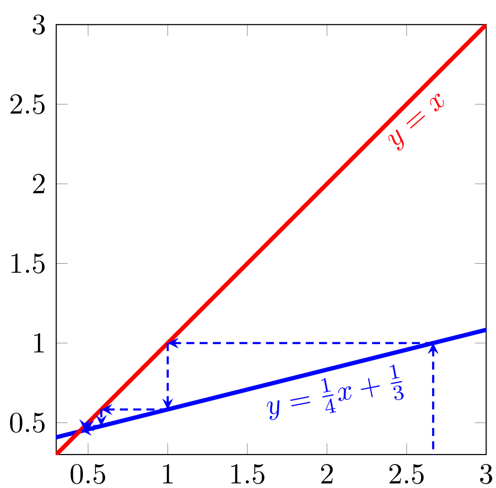


(a) Choose a value for a_1, and draw a diagram illustrating the convergence of $\{a_n\}$.

(b) Comparing this sequence with the previous one, how does the slope of the line defining the recurrence affect the rate of convergence.

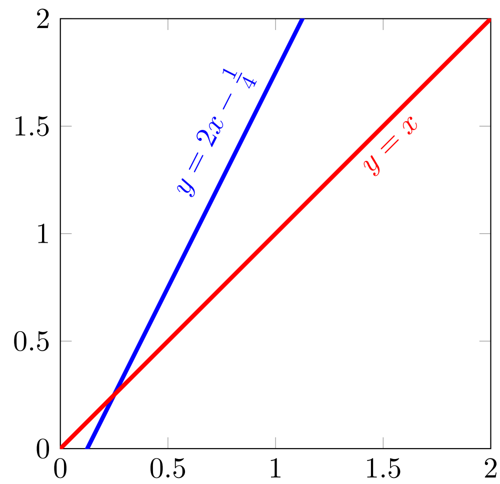
Solution.

(a)



(b) This larger slope causes faster convergence.

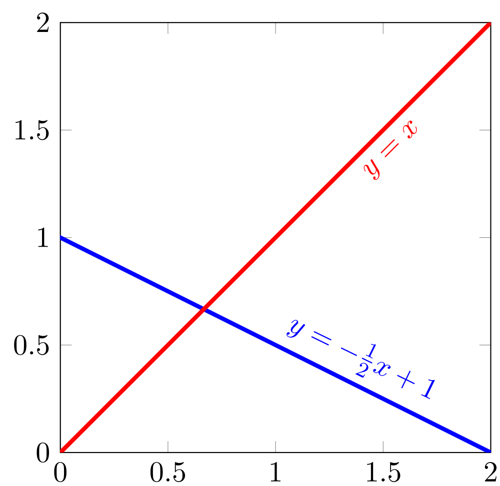
PROBLEM 3. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = 2a_n - \frac{1}{4}$.



Characterize the convergence behavior of the sequence $\{a_n\}$ for each choice of initial value a_1. (Are there any special choices for a_1?)

Solution. The sequence diverges for every initial value except $a_1 = 1/4$. When $a_1 = 1/4$, we get the constant sequence at $1/4$.

PROBLEM 4. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = -\frac{1}{2}a_n + 1$.



Characterize the convergence behavior of the sequence $\{a_n\}$ for each choice of initial value a_1.

Solution. The sequence converges to $2/3$ for every initial value. Note that the sequence in this case in not monotonic.

PROBLEM 5. Summarize the convergence behavior of a sequence defined by $a_1 \in \mathbb{R}$ and $a_{n+1} = ma_n + b$ for $m, b \in \mathbb{R}$.

(a) Consider the cases $|m| < 1$, $|m| > 1$, $m = -1$, and $m = 1$ separately. When does the sequence converge for all initial values? When does the sequence converge for only a special initial value (in which case the sequence is constant)?

(b) When is the sequence monotone?

(c) How does $|m|$ affect the rate of convergence or divergence?

Solution.

(a) When $|m| < 1$, the sequence converges for all initial values. It is constant when the initial value corresponds to the point in which $y = mx + b$ meets the line $y = x$, i.e., when $a_1 = ma_1 + b$. This happens when $a_1 = b/(1 - m)$.

When $|m| > 1$ the sequence diverges for all initial values except when $a_1 = b/(1-m)$. With this initial value, we get a constant sequence, as discussed above.

Next, consider the case $m = 1$. So the recursion is $a_{n+1} = a_n + b$ and the sequence is:

$$a_1, a_1 + b, a_1 + 2b, a_1 + 3b, \ldots$$

which converges if and only if $b = 0$, in which case we get a constant sequence.

Finally, consider the case where $m = -1$. The sequence in that case is

$$a_1,\ -a_1 + b,\ a_1,\ -a_1 + b,\ a_1,\ \ldots$$

Convergence occurs exactly when the initial value is $b/2$, which is $b/(1 - m)$ for the case $m = -1$.

(b) The sequence is monotone when $m > 0$ or in those special cases in which it is constant.

(c) The rate of convergence or divergence decreases as $|m|$ gets close to 1 (for generic initial conditions).

Problem 6. (Extra time.) Let $a_1 = 1$, and for $n \geq 1$, let $a_{n+1} := \frac{1}{2}a_n + \frac{1}{3}$. From the data shown below, it looks like the sequence $\{a_n\}$ is monotone decreasing and converging to $2/3$.

$$\begin{array}{llll} a_1 = 1.000000\ldots, & a_2 = 0.833333\ldots, & a_3 = 0.750000\ldots, & a_4 = 0.708333\ldots, \\ a_5 = 0.687500\ldots, & a_6 = 0.677083\ldots, & a_7 = 0.671875\ldots, & a_8 = 0.669271\ldots, \\ \ldots\quad , & a_{20} = 0.666667\quad , & \ldots & \end{array}$$

(a) Prove that $\{a_n\}$ is bounded below by $2/3$.

(b) Prove that $\{a_n\}$ is monotone decreasing.

(c) Thus, by the MCT, the sequence converges. Find its limit.

Solution.

(a) We will prove this by induction. For the base case, we have $a_1 = 1 \geq 2/3$. Suppose that $a_n \geq 2/3$ for some $n \geq 1$. Then

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{3} \geq \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{3} = \frac{2}{3}.$$

The result follows by induction.

(b) We will prove this by induction. For the base case, we have $a_1 = 1 \geq 5/6 = a_2$. Suppose that $a_n \geq a_{n+1}$ for some $n \geq 1$. Then

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{3} \geq \frac{1}{2}a_{n+1} + \frac{1}{3} =: a_{n+2}.$$

The result follows by induction.

(c) Say $\lim_{n\to\infty} a_n = a$. Then

$$\begin{aligned} a_{n+1} = \frac{1}{2}a_n + \frac{1}{3} \;&\Rightarrow\; \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{2}a_n + \frac{1}{3}\right) \\ &\Rightarrow\; a = \frac{1}{2}a + \frac{1}{3} \\ &\Rightarrow\; a = \frac{2}{3}. \end{aligned}$$