PROBLEM 1. Find the limit of the sequence $\left\{\frac{3n^2-5}{n^2-3n+2}\right\}$ using our limit theorems (i.e., without using an ε-N argument). Justify each step.

Solution. Using our limit theorems,

$$\begin{aligned}\lim_{n\to\infty}\frac{3n^2-5}{n^2-3n+2} &= \lim_{n\to\infty}\frac{\frac{1}{n^2}(3n^2-5)}{\frac{1}{n^2}(n^2-3n+2)}\\ &= \lim_{n\to\infty}\frac{3-\frac{5}{n^2}}{1-\frac{3}{n}+\frac{2}{n^2}}\\ &= \frac{\lim_{n\to\infty}3+(\lim_{n\to\infty}(-5))\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}{\lim_{n\to\infty}1+(\lim_{n\to\infty}(-3))\left(\lim_{n\to\infty}\frac{1}{n}\right)+(\lim_{n\to\infty}2)\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}\\ &= \frac{3-5\cdot 0^2}{1-3\cdot 0+2\cdot 0^2}\\ &= 3.\end{aligned}$$

PROBLEM 2. We have shown that $\lim_{n\to\infty}\frac{\sin(n)}{n}=0$. Use this result along with our limit theorems to find the limit of the sequence $\left\{\frac{\sin(n)}{n^2-n+1}\right\}$ justifying each step.

Solution. Using our limit theorems,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sin(n)}{n^2-n+1} &= \frac{\lim_{n\to\infty}\frac{1}{n^2}\sin(n)}{\lim_{n\to\infty}\frac{1}{n^2}(n^2-n+1)}\\ &= \frac{\lim_{n\to\infty}\frac{\sin(n)}{n^2}}{\lim_{n\to\infty}\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}\\ &= \frac{\left(\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)}{(\lim_{n\to\infty}1)+(\lim_{n\to\infty}(-1))\left(\lim_{n\to\infty}\frac{1}{n}\right)+\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}\\ &= \frac{0\cdot 0}{1-1\cdot 0+0^2}\\ &= 0.\end{aligned}$$

PROBLEM 3. State whether each of the following statements is true or false (with proof or concrete counterexample):

(a) If $\{a_n\}$ and $\{b_n\}$ both diverge, then $\{a_n+b_n\}$ diverges.

(b) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n+b_n\}$ diverges.

Solution.

(a) False. Consider $a_n = n$ and $b_n = -n$.

(b) True. Suppose $\{a_n\}$ converges. Then by the limit theorems, if $\{a_n + b_n\}$ converges, it follows that

$$\lim_{n\to\infty} (a_n + b_n) - \lim_{n\to\infty} a_n$$

exists and equals $\lim_{n\to\infty} b_n$. So $\{b_n\}$ would have to converge.

PROBLEM 4. Let $k \in \mathbb{N}_{>0}$. Find, with proof, the limit of the sequence $\left\{\left(\frac{n+1}{n}\right)^k\right\}$.

Solution. We find

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^k &= \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^k \\
&= \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\right)^k \\
&= \left(\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}\right)^k \\
&= (1+0)^k \\
&= 1^k \\
&= 1.
\end{aligned}$$

PROBLEM 5. Suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Review the proof that

$$\lim_{n\to\infty} (s_n + t_n) = s + t.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, there exists $N_s \in \mathbb{R}$ such that $n > N_s$ implies $|s - s_n| < \varepsilon/2$. Similarly, there exists $N_t \in \mathbb{R}$ such that $n > N_t$ implies $|t - t_n| < \varepsilon/2$. Let $N := \max\{N_s, N_t\}$. Then $n > N$ implies both $|s - s_n| < \varepsilon/2$ and $|t - t_n| < \varepsilon/2$, simultaneously. Using the triangle inequality, it follows that if $n > N$,

$$|(s+t) - (s_n + t_n)| = |(s - s_n) + (t - t_n)| \leq |s - s_n| + |t - t_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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