# Dynamical Systems

**self-mapping** of a set $S$: a function $f\colon S \to S$.

**$n$-th iterate** of $s$ under $f$:

$$f^n(s) := \begin{cases} s & \text{if } n = 0 \\ f(f^{n-1}(s)) & \text{if } n > 0. \end{cases}$$

**orbit** of $s$ under $f$: $\mathrm{Orb}_f(s) := \{s, f(s), f^2(s), f^3(s), \ldots\}$ = the iterates of $s$ under $f$.

**fixed points** of $f$: $\mathrm{Fix(f)} := \{s \in S : f(s) = s\}$.

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In the following problems, let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$. Our goal is to understand the orbits of $f$.

Problem 1. What is the orbit of 0 under $f$? What is the orbit of $-1$?

Problem 2. What are the first four iterates of $\frac{1}{2}$, i.e., $f^0(1/2), f^1(1/2), f^2(1/2), f^3(1/2)$? (You do not need to evaluate.)

Problem 3. Label the 12 dots in Figure 1 using the notation $f^i(1/2)$.

Problem 4. What are the fixed points of $f$? How can you picture these in Figure 1?

Problem 5. Draw a picture as in Figure 1 with an initial point just to the left of the positive fixed point. See Figure 2.

Problem 6. Use induction to prove that if $x \in [-1, 0]$, then $f^n(x) \in [-1, 0]$ for all $n \geq 0$. (You may use standard facts about real numbers.)

Problem 7. Show that if $x \in [-1, 1]$, then $f(x) \in [-1, 0]$ for all $n \geq 1$. (What if $x \in [0, 1]$?)

**Facts.** Let $\alpha$ denote the positive fixed point of $f$. Then:

- If $x \in (\alpha, \infty)$, then the iterates of $x$ increase without bound. (So the orbit of $x$ is unbounded.)
- If $x \in (-\infty, -\alpha)$, then $f(x) \in (\alpha, \infty)$.
- If $x \in (1, \alpha)$, then $f^n(x)$ decreases until an iterate is in $[0, 1]$.
- If $x \in (-\alpha, -1)$, then $f(x) \in (0, \alpha)$.

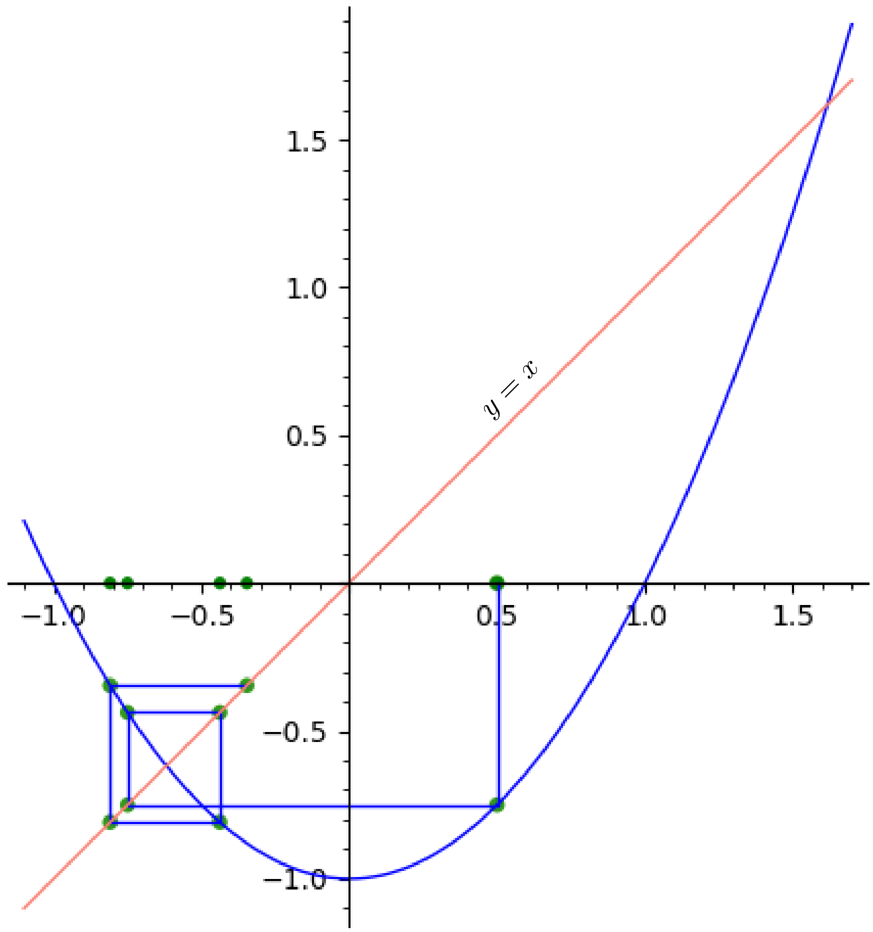


FIGURE 1. Visualizing the dynamical system determined by $f(x) = x^2 - 1$.

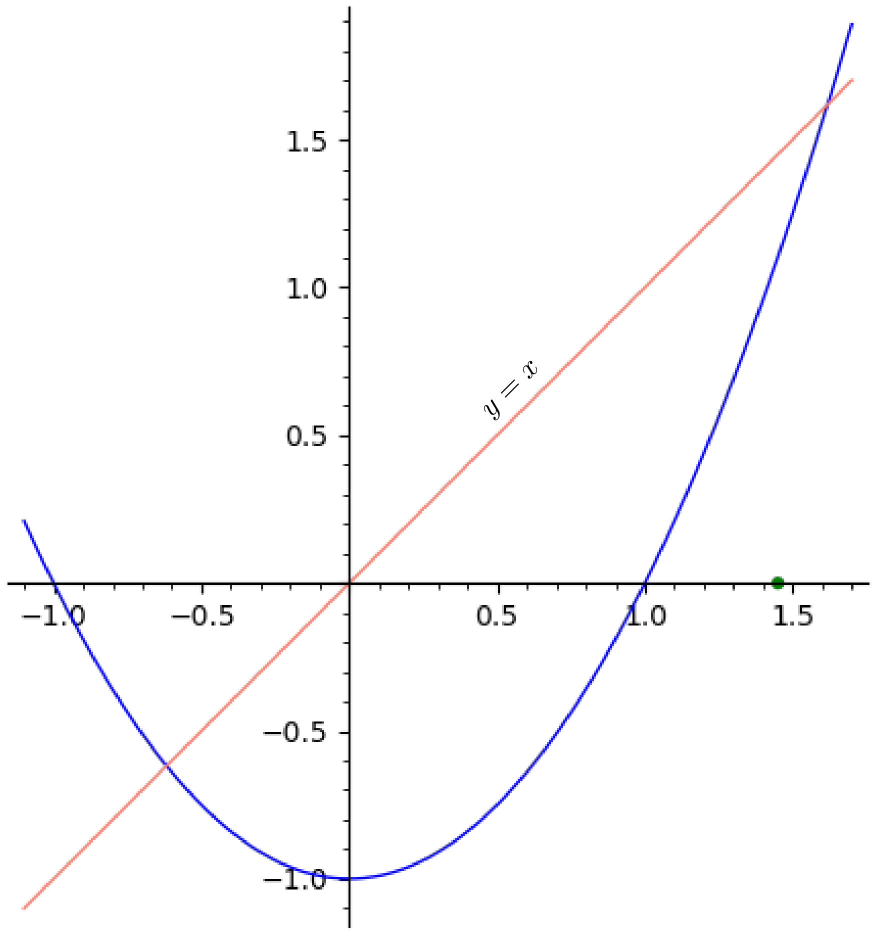


FIGURE 2. Draw a picture as in Figure 1 with an initial point just to the left of the positive fixed point.