Dynamical Systems

self-mapping of a set S: a function $f\colon S \to S$.

n**-th iterate** of s under f:

$$f^n(s) := \begin{cases} s & \text{if } n = 0 \\ f(f^{n-1}(s)) & \text{if } n > 0. \end{cases}$$

orbit of s under f: $\mathrm{Orb}_f(s) := \{s, f(s), f^2(s), f^3(s), \ldots\}$ = the iterates of s under f.

fixed points of f: $\mathrm{Fix(f)} := \{s \in S : f(s) = s\}$.

In the following problems, let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$. Our goal is to understand the orbits of f.

PROBLEM 1. What is the orbit of 0 under f? What is the orbit of -1?

SOLUTION: We have $f(0) = -1$ and $f(f(0)) = f(-1) = 0$. Therefore, the orbit of 0 is $\mathrm{Orb}_f(0) = \{-1, 0\}$, and the orbit of -1 is $\mathrm{Orb}_f(-1) = \{-1, 0\}$.

PROBLEM 2. What are the first four iterates of $\frac{1}{2}$, i.e., $f^0(1/2), f^1(1/2), f^2(1/2), f^3(1/2)$? (You do not need to evaluate.)

SOLUTION: We have

$$\begin{aligned} f^0(1/2) &= 1/2 \\ f^1(1/2) &= (1/2)^2 - 1 = -3/4 \\ f^2(1/2) &= f(-3/4) = (-3/4)^2 - 1 = -7/16 \\ f^3(1/2) &= f(-7/16) = (-7/16)^2 - 1 = -207/256. \end{aligned}$$

PROBLEM 3. Label the 12 dots in Figure 1 using the notation $f^i(1/2)$.

SOLUTION: See Figure 1.

PROBLEM 4. What are the fixed points of f? How can you picture these in Figure 1?

Solution: We have $x = f(x) = x^2 - 1$, or $x^2 - x - 1 = 0$. The two solutions to this equation are

$$\frac{1 \pm \sqrt{5}}{2}.$$

We can visualize these values as the x-coordinates of the point of intersection of the line $y = x$ with the graph of f in Figure 1.

Problem 5. Draw a picture as in Figure 1 with an initial point just to the left of the positive fixed point. See Figure 2.

Solution: See Figure 2.

Problem 6. Use induction to prove that if $x \in [-1, 0]$, then $f^n(x) \in [-1, 0]$ for all $n \geq 0$. (You may use standard facts about real numbers.)

Solution: Let $x \in [-1, 0]$. The base case holds since $f^0(x) = x \in [-1, 0]$. Suppose that $a := f^n(x) \in [-1, 0]$ for some $n \geq 0$. Then

$$f^{n+1}(x) = f(f^n(x)) = f(a) = a^2 - 1.$$

Since $a \in [-1, 0]$, it follows that $a^2 \in [0, 1]$. (Details: We have $0 \leq -a < 1$, which implies that $0 \leq (-a)^2 \leq 1$.) Therefore, $f(a) = a^2 - 1 \in [-1, 0]$. The result follows by induction.

Problem 7. Show that if $x \in [-1, 1]$, then $f(x) \in [-1, 0]$ for all $n \geq 1$.

Solution: If $x \in [-1, 1]$, then $x^2 \in [0, 1]$, and hence, $f(x) = x^2 - 1 \in [-1, 0]$. The result then follows from the previous problem.

Facts. Let α denote the positive fixed point of f. Then:

» If $x \in (\alpha, \infty)$, then the iterates of x increase without bound. (So the orbit of x is unbounded.)
» If $x \in (-\infty, -\alpha)$, then $f(x) \in (\alpha, \infty)$.
» If $x \in (1, \alpha)$, then $f(^n)(x)$ decreases until an iterate is in $[0, 1]$.
» If $x \in (-\alpha, -1)$, then $f(x) \in (0, \alpha)$.

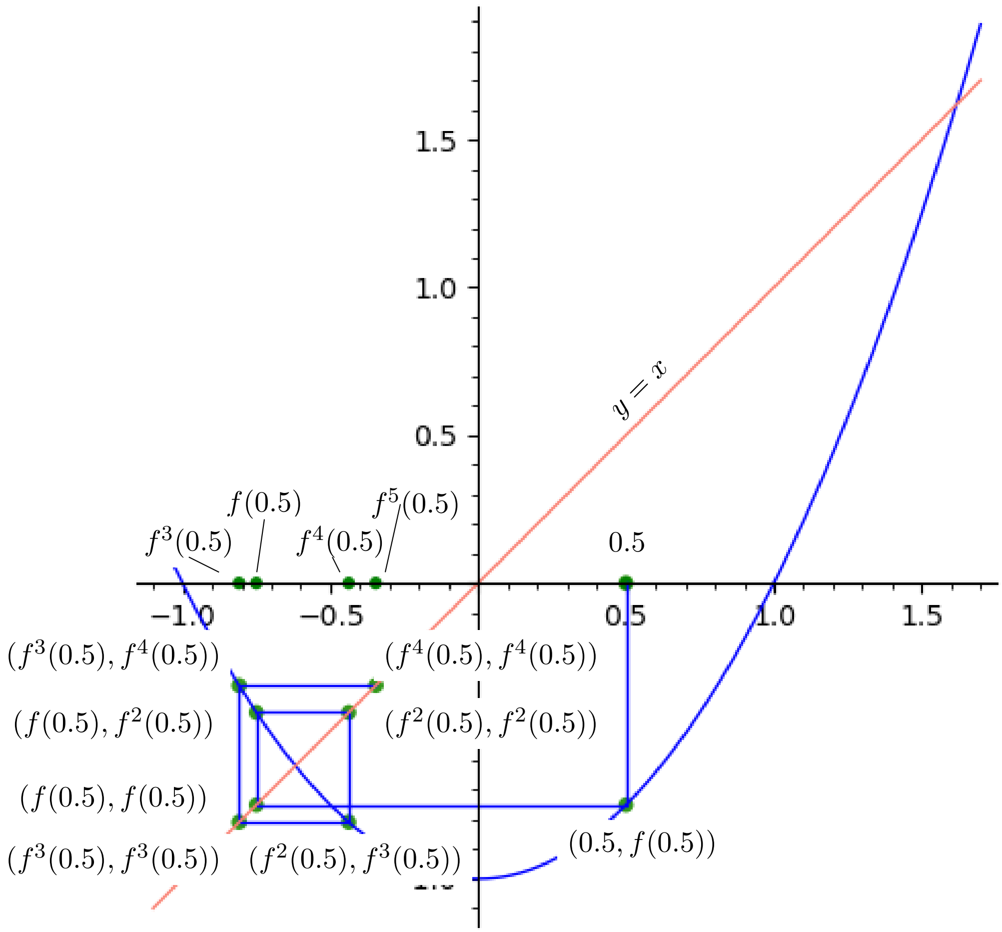

FIGURE 1. Visualizing the dynamical system determined by $f(x) = x^2 - 1$.

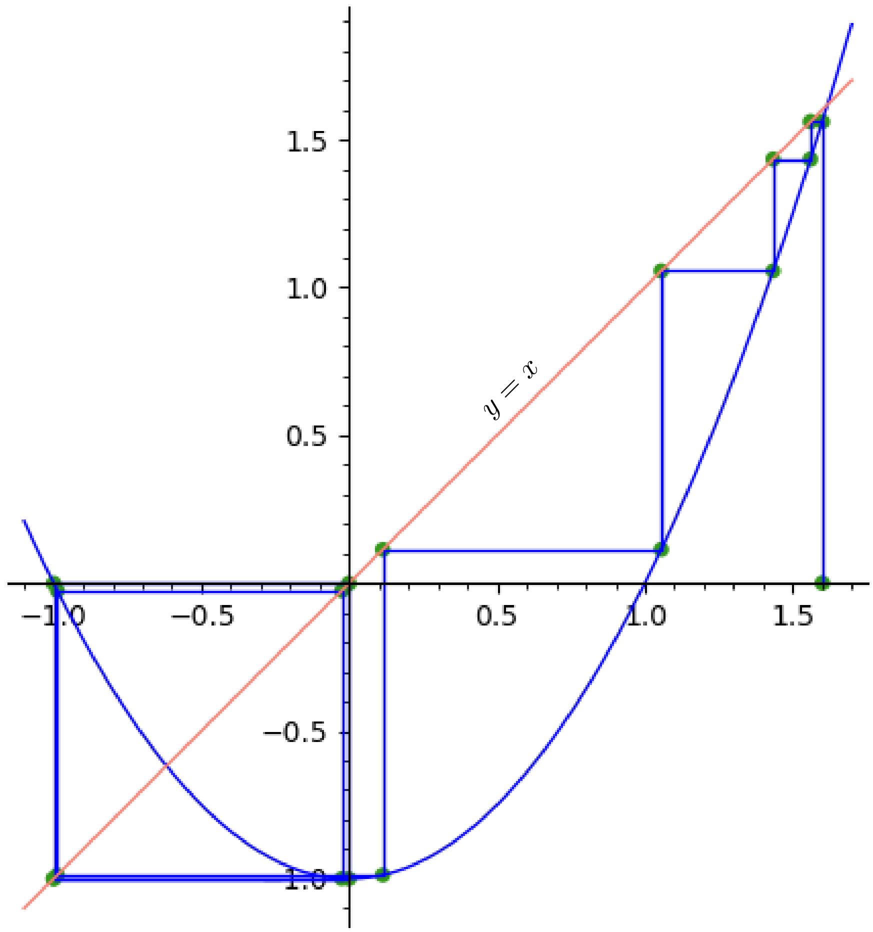

FIGURE 2. Draw a picture as in Figure 1 with an initial point just to the left of the positive fixed point.