Dynamical Systems

Let $S = \mathbb{R}$ or $\mathbb{C}$, and let $f\colon S \to S$. The *filled Julia set* for f is

$$K(f) = \{z \in S\colon \mathrm{Orb}_f(z) \text{ is bounded}\}.$$

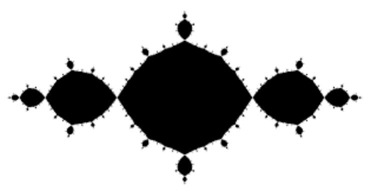
Thus, $K(f)$ is the set of points $z \in S$ whose iterates are bounded: there exists a real number r such that $|f^n(z)| \le r$ for all $n \ge 0$. The *Julia set*, denoted $J(f)$, is the boundary[1] of $K(f)$.

Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 1$. We saw last time that $K(f) = [-\alpha, \alpha]$ where $\alpha = \frac{1+\sqrt{5}}{2}$. Thus, $J(f)$ consists of the two endpoints: $J(f) = \{-\alpha, \alpha\}$.

Problem 1. What are the filled Julia set and the Julia set for the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$?

Problem 2. What are the filled Julia set and the Julia set for the function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$?

Problem 3. Consider the filled Julia set $K(f)$ for the function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 1$. The set $K(f) \subseteq \mathbb{C}$ is pictured below:



(a) What is the horizontal line segment running through the middle (along the real axis)?

(b) Is $i \in K(f)$? What about $i/2$? (Hint: use that fact that you know something about the filled Julia set for f restricted to the real numbers. Note: $1.61 \le (1 + \sqrt{5})/2 \le 1.62$.)

Problem 4. Let $c \in \mathbb{C}$ and consider the function $f_c\colon \mathbb{C} \to \mathbb{C}$ defined by $f_c(z) = z^2 + c$. (For instance, $f_{-1}(z) = z^2 - 1$.) Show that $K(f_c)$ is symmetric about the origin.

Problem 5. Go to https://www.marksmath.org/visualization/julia_sets/. There are two copies of $\mathbb{C}$ pictured on that page. Clicking a point on the left side selects a point $c \in \mathbb{C}$, and the number c is displaying in a box underneath. You can choose c without clicking by entering it in this box. The right side then shows the Julia set for $f_c(z) = z^2 + c$.

[1] If X is a subset of a topological space, the *closure* of X, denoted $\overline{X}$, is the smallest closed set containing K. It is the intersection of all closed set containing X. The *boundary* of X is the intersection of the closure of X and the closure of the complement of X. Example: the closure of an open ball in $\mathbb{C}$ is a circle.

(a) Enter the point $c = 0$ to see the Julia set for $f_{-1}(z) = z^2$. (You will see the point displayed in the set on the left.)

(b) Enter the point $c = -1$ to see the Julia set for $f_{-1}(z) = z^2 - 1$.

(c) What happens as you click points along the real axis going from 0 to -1?

(d) Hit "Clear" to erase the Julia sets drawn so far. The shape pictured in the left is the Mandelbrot set, M. It is the set of points $c \in \mathbb{C}$ such that the iterates of 0 under the mapping $f_c(z) = z^2 + c$ are bounded, i.e., $0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$ is bounded. What distinguishes Julia sets for $c \in M$ and $c \notin M$?

Problem 6. Explain why $K(f_c)$ is symmetric about the real axis.

Problem 7. Prove that for all $c \in \mathbb{C}$, we have $K(f_c) \neq \emptyset$.